Engineering Tripos, Part IIB

4C2 Designing with Composites, 2019

Solutions   Michael Sutcliffe

1 (a) The axial and transverse stiffnesses are given by

$$E_1 = \left[fE_\mathrm{f} + (1-f)E_\mathrm{m} \right] = 0.6(232) + 0.4(3) = 140.4 \text{ GPa}$$

To predict the axial stiffness of an aligned composite, we represent both constituents, the matrix and the fibres, as parallel slabs welded together, with thicknesses in proportion to their volume fraction. When the stress is applied parallel to the fibres, an equal strain condition applies. Since the slabs are bonded together, both the fibres and the matrix will stretch by the same amount in this direction, i.e. they will have equal strains (matrix and fibre in "parallel").

When the stress is applied transverse to the fibre direction, the stiffness can be predicted using an equal stress assumption. The fibres and the matrix carry the same stress, like springs in series. The stiffness of the composite is much lower in this case, since the (compliant) matrix is not shielded from carrying stress to the same degree as for axial loading (there is no transfer of stress to the stiffer constituent).

$$E_2 = \left[\frac{f}{E_\mathrm{f}} + \frac{(1-f)}{E_\mathrm{m}} \right]^{-1} = \left[\frac{0.6}{232} + \frac{0.4}{3} \right]^{-1} = 7.36 \,\text{GPa}$$

(b) (i)

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \Rightarrow \nu_{21} = 0.016$$

Calculate $[Q]$ in principal material axes (1, 2)

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}} = \frac{140.4}{1-0.3\times 0.016} = 141.1 \text{ GPa}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}} = \frac{7.36}{1-0.3\times 0.016} = 7.40 \text{ GPa}$$

$$Q_{12} = \frac{\nu_{12} E_2}{1-\nu_{12}\nu_{21}} = \frac{0.3\times 7.36}{1-0.3\times 0.016} = 2.22 \text{ GPa}$$

$$Q_{66} = G_{12} = 6.92 \text{ GPa} \qquad Q_{16} = Q_{26} = 0$$

$$[Q] = \begin{bmatrix} 141.1 & 2.22 & 0 \\ 2.22 & 7.40 & 0 \\ 0 & 0 & 6.92 \end{bmatrix} \text{ GPa}$$

(c) (i) Calculate the transformed stiffness matrix $[\overline{Q}]$ in the global x-y axes.

The transformed lamina stiffness matrix $[\overline{Q}]$ for the 0° plies is unchanged

The transformed stiffness matrix for the +60° plies is given by

$$\left(\overline{Q}_{11}\right)_{60^\circ} = 141.4\,c^4 + 7.40\,s^4 + 2\left(2.22 + 2\times 6.92\right)s^2c^2 = 19.03\text{ GPa}$$

$$\left(\overline{Q}_{12}\right)_{60^\circ} = \left(141.4 + 7.40 - 4\times 6.92\right)s^2c^2 + 2.22\left(c^4 + s^4\right) = 24.05\text{ GPa}$$

$$\left(\overline{Q}_{22}\right)_{60^\circ} = 141.4\,s^4 + 7.40\,c^4 + 2\left(2.22 + 2\times 6.92\right)s^2c^2 = 85.84\text{ GPa}$$

$$\left(\overline{Q}_{66}\right)_{60^\circ} = \left(141.4 + 7.40 - 2\times 2.22 - 2\times 6.92\right)s^2c^2 + 6.92\left(s^4 + c^4\right) = 28.74\text{ GPa}$$

where $c = \cos 60 = 0.5,\ s = \sin 60 = 0.866$

No need to calculate Q_{16} and Q_{26} as these will cancel out.

$$\left[\overline{Q}\right]_{60^\circ} = \begin{bmatrix} 19.03 & 24.05 & - \\ 24.05 & 85.84 & - \\ - & - & 28.74 \end{bmatrix}\text{ GPa}$$

The relevant parts of the transformed lamina stiffness matrix $[\overline{Q}]$ for the -60° plies are the same.

Set t (=0.25 $\times 10^{-3}$ m) for lamina thickness

$$A_{11} = \left[\left(\overline{Q}_{11}\right)_{+60} + \left(\overline{Q}_{11}\right)_{0} + \left(\overline{Q}_{16}\right)_{-60}\right]\cdot 16t = 716.5\text{ MN m}^{-1}$$

$$A_{12} = \left[\left(\overline{Q}_{12}\right)_{+60} + \left(\overline{Q}_{12}\right)_{0} + \left(\overline{Q}_{12}\right)_{-60}\right]\cdot 16t = 201.3\text{ MN m}^{-1}$$

$$A_{22} = \left[\left(\overline{Q}_{22}\right)_{+60} + \left(\overline{Q}_{22}\right)_{0} + \left(\overline{Q}_{22}\right)_{-60}\right]\cdot 16t = 716.5\text{ MN m}^{-1}$$

$$A_{16} = \left[\left(\overline{Q}_{16}\right)_{+60} + \left(\overline{Q}_{16}\right)_{0} + \left(\overline{Q}_{26}\right)_{-60}\right]\cdot 16t = 0$$

$$A_{26} = \left[\left(\overline{Q}_{26}\right)_{+60} + \left(\overline{Q}_{26}\right)_{0} + \left(\overline{Q}_{26}\right)_{-60}\right]\cdot 16t = 0$$

$$A_{66} = \left[\left(\overline{Q}_{66}\right)_{+60} + \left(\overline{Q}_{66}\right)_{0} + \left(\overline{Q}_{66}\right)_{-60}\right]\cdot 16t = 257.6\text{ MN m}^{-1}$$

$$[A] = \begin{bmatrix} 716.5 & 201.3 & 0 \\ 201.3 & 716.5 & 0 \\ 0 & 0 & 257.6 \end{bmatrix}\text{ MNm}^{-1}$$

Since A_{16}=A_{26}=0. the laminate is balanced. This means that the laminate as whole does not exhibit any tensile-shear interactions. Tensile-shear interactions are tensile strains arising from applied shear stresses and visa versa and result in in-plane distortion of the laminate.

Furthermore, because the laminate is quasi-isotropic (the laminae are oriented at the same angle relative to adjacent laminae), [A] has the following form

$$[A]=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & (A_{11}-A_{12})/2 \end{bmatrix}$$

The stacking sequence is symmetrical, hence [B] = 0. This means that there is no bending-stretching coupling in this laminate. In-plane loads do not generate bending and twisting curvatures which cause out-of-plane distortion. Also, bending and twisting does not produce an extension of the middle surface.

(ii) By symmetry $\gamma_{xy} = 0$.

$$\frac{P\pi d^2}{4} = N\pi d \Rightarrow N_x = N_y = \frac{Pd}{4}$$

$\therefore N_x = N_y = 0.5P = 0.5$ MN m^{-1} (where d = 2 m and P = 1 bar = 1 MPa)

$N_{xy} = 0$

$$\begin{pmatrix} \varepsilon_{\mathrm{x}} \\ \varepsilon_{\mathrm{y}} \\ \gamma_{\mathrm{xy}} \end{pmatrix} = [A]^{-1} \begin{pmatrix} \frac{pd}{4} \\ \frac{pd}{4} \\ 0 \end{pmatrix}$$

$$[A]=\begin{bmatrix} 716.5 & 201.3 & 0 \\ 201.3 & 716.5 & 0 \\ 0 & 0 & 257.6 \end{bmatrix} \mathrm{MNm}^{-1}$$

$$\begin{pmatrix} \varepsilon_{\mathrm{x}} \\ \varepsilon_{\mathrm{y}} \\ \gamma_{\mathrm{xy}} \end{pmatrix} = [A]^{-1} \begin{pmatrix} N_{\mathrm{x}} \\ \mathrm{N_y} \\ \mathrm{N_{xy}} \end{pmatrix} = \begin{bmatrix} 0.001515 & -0.0004258 & 0 \\ -0.0004258 & 0.001515 & 0 \\ 0 & 0 & 0.003882 \end{bmatrix} \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0.545\times10^{-3} \\ 0.545\times10^{-3} \\ 0 \end{pmatrix}$$

$\varepsilon_{\mathrm{x}} = \varepsilon_y$, hence vessel expands equally in all directions (i.e. remains a sphere).

[A standard question attempted by all candidates with a correspondingly high average. Since laminate plate theory is at the heart of the course, the good marks for this question are reassuring evidence that all students have mastered this aspect well.]

2 (a) The idea is to check the weakest link assuming 0°, 45, 90° plies are all included.

By calculating the laminate ε based on laminate plate theory a laminate failure point can be estimated without calculating individual ply stresses. Though somewhat inaccurate and limited by applicability of ε allowables used, it gives a useful starting point. Strain allowables given on the datasheet are typically determined by transverse failure of off-axis plies. An exception is kevlar which has a low failure ε in compression

(b) (i) 2mm thickness ⇒ 16 plies {often omitted}

Maximise torsional stiffness ⇒ maximise G_{xy}.

Could we leave out off-axis plies (0, 90) just having $[(\pm 45)_4]_s$ which maximises G_{xy}. This is the preferred choice, but if following 'rule of thumb' include some 0° plies, eg $[(\pm 45) 0_2 (\pm 45)_2]_s$

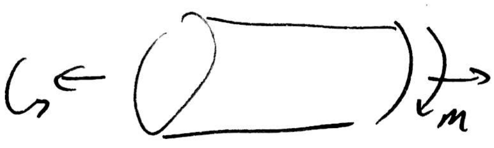
(ii) Need to maximise E_x.

Could choose $(0)_{16}$ but this will be prone to splitting. Better to include some 45's (90's will be hard to consolidate in tube) eg. $[(\pm 45) 0_6]_s$ on outside for impact resistance)

[Candidates were mostly OK with the above, though failing to discuss options for off-axis plies. Less good with (b)(iv) where the analysis was not done well.]

(iii) We simply want to use +-45s. I think that it would be fine in this case to omit 0s as splitting is not so likely in this case with plies in the two orthogonal directions.

2 (b) (iv)



$Q = N_{xy} \cdot R \cdot 2\pi R$

$\frac{\sigma}{y} = \frac{M}{I}, \; N_x = \sigma_x t, \; I = \pi R^3 t$

$\rightarrow N_x = \frac{M R t}{\pi R^3 t} = \frac{M}{\pi R^2}$

Using ε allowables: $\varepsilon_x = \frac{N_x}{t E_x} = \frac{M}{t \pi R^2 E_x}$

$\gamma_{xy} = \frac{N_{xy}}{t G_{xy}} = \frac{Q}{2\pi R^2 t G_{xy}}$

For GFRP $e^+ = 0.3\,\%$ (critical c.f. $\bar{\varepsilon}$), $e_{LT} = 0.5\%$

To maximise strength ~~pat~~ choose laminate so fails by e^+ and e_{LT} at the same time.

i.e. $\frac{\varepsilon_x}{\gamma_{xy}} = \frac{e^+}{e_{LT}} = \frac{0.3}{0.5} = \frac{M / t\pi R^2 E_x}{Q / 2\pi R^2 t G_{xy}} \Rightarrow \frac{G_{xy}}{E_x} = 0.3$

Expect plenty of 45s, 0s, omit 90s, though ε allowable won't be strictly applicable.

Try $0^\circ : 45^\circ = 50:50 \Rightarrow G_{xy} = 8.8$ GPa, $E_x = 30$ GPa, $\frac{G_{xy}}{E_x} = 0.29$ ✓

[Could iterate to this answer if first guess wasn't good.]

$\Rightarrow [(\pm 45)_2\, 0_4]_s$ is a good choice, balanced symmetric.

(c) Manufacturing - need to consider eg via pultrusion or filament winding

Testing - prototyping, check features including joints

Costs - need to check carefully

More sophisticated failure analysis

Environmental, fatigue

Shape optimisation

impact / damage assesment.

3(a) Max σ & ε criteria -

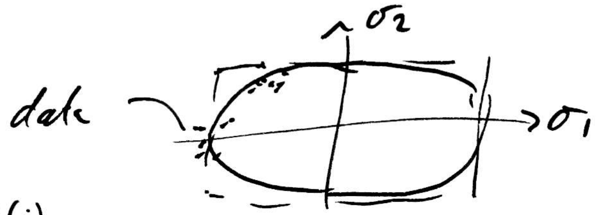
- compare σ/ε in different directions with critical values
- simple to apply
- less accurate
- differ due to Poisson's ratio effects

Quadratic (Tsai-Hill, Tsai-Wu)

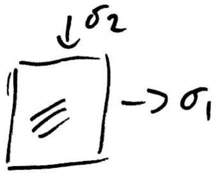
- include coupling terms so can be more accurate
- need to collect additional data

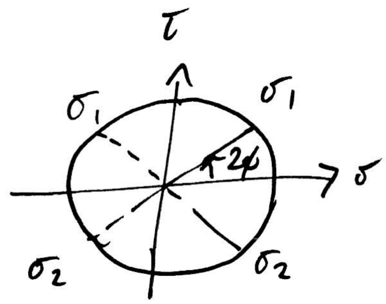
In general need to take into account scatter in data, plus 3D loading cases.

Can represent failure criteria as failure surfaces



(b)(i)





I think this step, identifying signs of stresses, is helpful

Note same loading for $\phi = -\phi$

For $90 > \phi > 45°$, same loading as $(45 - \phi)$, but signs reversed

For $45° > \phi > 0°$, σ_1 tensile, σ_2 compressive

$$\sigma_1 = \sigma(c^2 - s^2), \quad \sigma_2 = \sigma(s^2 - c^2), \quad \sigma_{12} = -2\sigma sc \qquad \begin{pmatrix} c = \cos\phi \\ s = \sin\phi \end{pmatrix}$$

Tsai-Hill $\dfrac{\sigma_1^2}{S_L^2} - \dfrac{\sigma_1\sigma_2}{S_L^2} + \dfrac{\sigma_2^2}{S_T^2} + \dfrac{\sigma_{12}^2}{S_{LT}^2} = 1$

or use double angle formulae and solve for ϕ

For $\phi = 5.6°$, $s = 0.0961$, $c = 0.9956$, $\sigma_1 = 0.9823\sigma$, $\sigma_2 = -\sigma_1$, $\sigma_{12} = -0.1876\sigma$

$$T\text{-}H \Rightarrow \underset{(S_L^+)}{\frac{0.9823^2}{400}} + \frac{0.9223^2}{400} + \underset{(S_T^-)}{\frac{0.9823^2}{4}} + \frac{0.1876^2}{9} = \frac{\Sigma^2}{\sigma^2} = 0.2499$$

$$\Rightarrow \sigma = 2\Sigma$$

this check often missed

For $\phi = 0$, $\sigma_1 = \sigma_2 = 0$. $TH \Rightarrow \frac{1}{400} + \frac{1}{400} + \frac{1}{4} = \frac{\Sigma^2}{\sigma^2} \Rightarrow \sigma = 1.98\Sigma$

So $\sigma(\text{failure}) < 2\Sigma$ for $-5.6° < \theta < 5.6°$

3 (b)(ii) Pure shear for $\phi = 65°$

As ϕ increases expect transverse tension in the range $45° < \phi < 90°$ and transverse compression in the range $0 < \phi < 45°$. } This logic often missed

→ need to compare shear failure with these transverse failure modes. At crossover $\frac{\sigma_2}{S_T} = \frac{\sigma_{12}}{S_{LT}}$

ϕ	25	30	35	45	55	60	65
s	.42	.5	.57				
c	.82	.75	.82				
σ_2	−0.64	−.5	−.34				
σ_{12}	−0.77	−.866	−.94				
$\left\lvert\frac{\sigma_2}{\sigma_{12}}\right\rvert$	0.84	0.58	0.366	0			

approx $\frac{2}{3}$ approx $\frac{1}{3}$

Transverse compression ← → Transverse tension

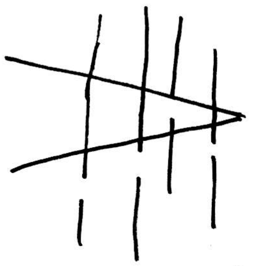
$\sigma_F \ \frac{S_T}{S_{LT}} = \frac{2}{3}$ $\frac{S_T}{S_{LT}} = \frac{1}{3}$

So expect shear failure for $30° < \phi < 55°$ approx

[Could use double angle formula to find exact values: $28° < \phi < 54°$]

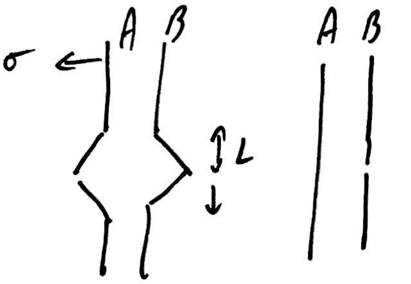
Candidates tended not to spot different failure modes before launching in to algebra.

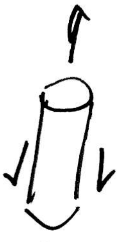
4(a)



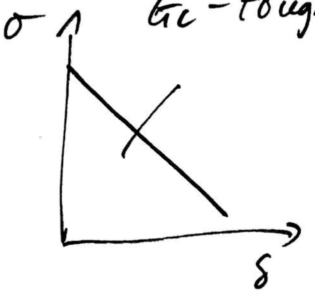
Bridging stress governed by embedded length and corresponding pull-out stress. Embedded length determined by statistics of fibre failure but the shear lag length for load transfer from a break gives an estimate.

Shear lag theory also gives an estimate of the pull-out stress

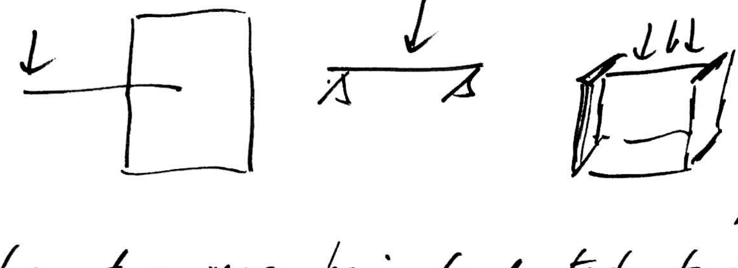




As fibres are pulled out bridging stress drops giving relationship between stress and crack opening δ and hence toughness Gc



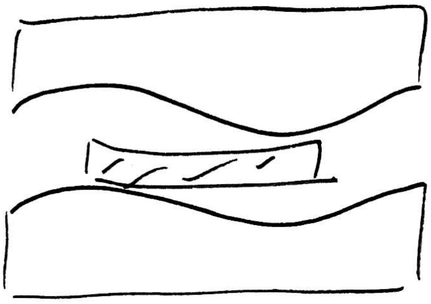
(b) Drop tests, notched tests and compression after impact tests are all standard to characterise impact. But difficult to



interpret and apply to real designs. Need to distinguish, low, moderate & high energy impacts

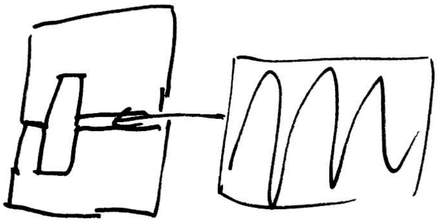
Also at a more basic level tests to assess delamination toughness such as double cantilever beam or edge-notched flexure could be used.
Laminate design - effect of off-axis plies, woven layers to prevent laminate splitting ← This part not answered well
Delamination affected by interlaminar toughness
Manufacturing differences can influence bonding, defects, porosity. quality of cure, which is important, along with QA and larger-scale tests.

4 (c)



Matched dies. Introduce charge. Could use sheet moulding compound.
Apply load to deform charge.
Relatively low cost.

Answered well



High-rate injection process using screw injector.
Low fibre fraction, only relatively short fibres survive.

(d) Cost - always important affecting material and process. eg boat and truck parts are cost sensitive so likely to be GFRP while the mast could be made from CFRP.

Process - depends on size and shape of component and production rate. Airbag housing needs to be high rate => injection process, though this will compromise mechanical performance. Deflector could be modest rate => RTM. Yacht mast needs maximise performance => autoclaving from hand or tape lay-up. Or perhaps pultrusion

Performance - again linked with cost. Boat could be relatively low performance but mast needs to be high quality

Environmental - need to consider eg. salt water attack and fatigue.

Mostly answered well. Part (d) tended to be a list of factors for each example without drawing out themes.