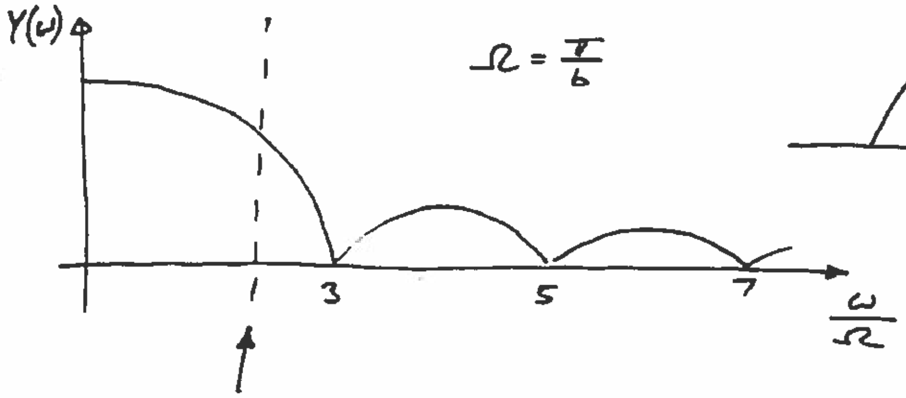
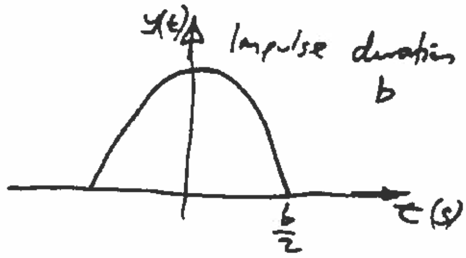
1 (a)



$\Omega = \frac{\pi}{b}$



Useful frequency $\omega_{useful} \sim 2\Omega = \frac{2\pi}{b}$ (rad s^{-1})

$\therefore f_{useful} \sim \frac{1}{b}$ (Hz)

Say $f_{useful} = 1000$ Hz $\quad \therefore \underline{\underline{b = 1 \text{ ms}}}$

$\Omega = \frac{\pi}{b} \approx 3000 \text{ rad s}^{-1} = \sqrt{\frac{k}{m}}$

$\therefore k = m\Omega^2 = 0.1 \times (3000)^2$

$= \underline{\underline{900000 \text{ N/m}}}$

[20%]

(b) Peak velocity = 3 ms^{-1}

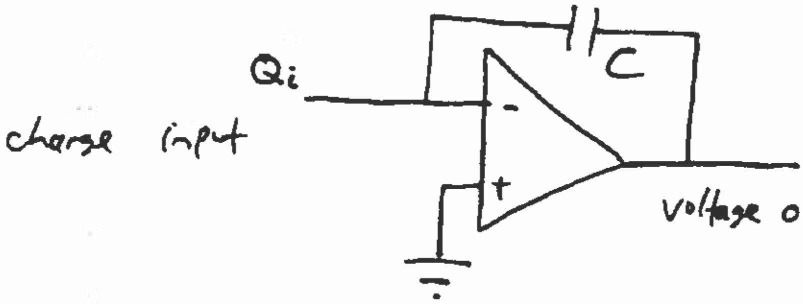
$\therefore$ Peak displacement $= \frac{v_{max}}{\Omega} = \frac{3}{3000} = 0.001 \text{ m}$

Peak force $= k\,x_{max} = 900$ N

$\therefore$ Peak charge output $= 5 \times 900 = 4500$ pC

which corresponds to say 4.5V in the datalogger

$\therefore$ Charge amp gain $= 10^{-3}$ V/pC or 1 V/nC



$C = 1 \text{ nF}$

$V_o = \frac{Q_i}{C}$

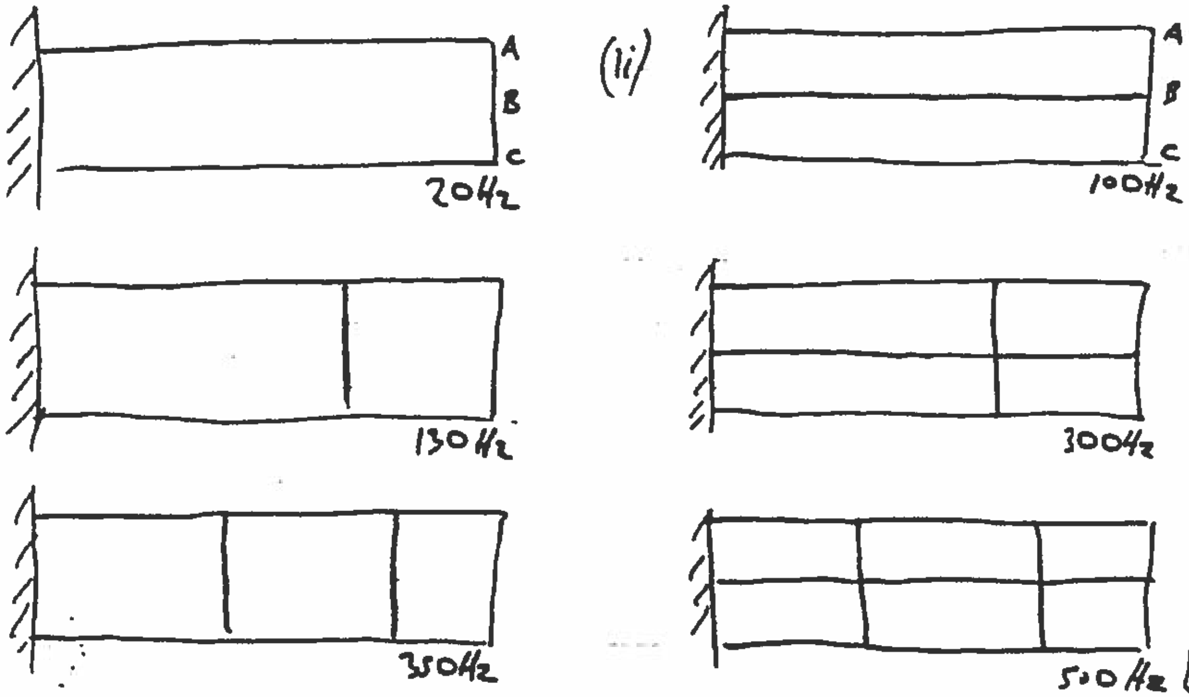
[20%]

1 cont.

(c) Want sampling rate well above any response frequency. The hammer excites usefully up to 1000Hz but there will be input measurable up to say 5000Hz. Nyquist criterion requires that we log at twice this

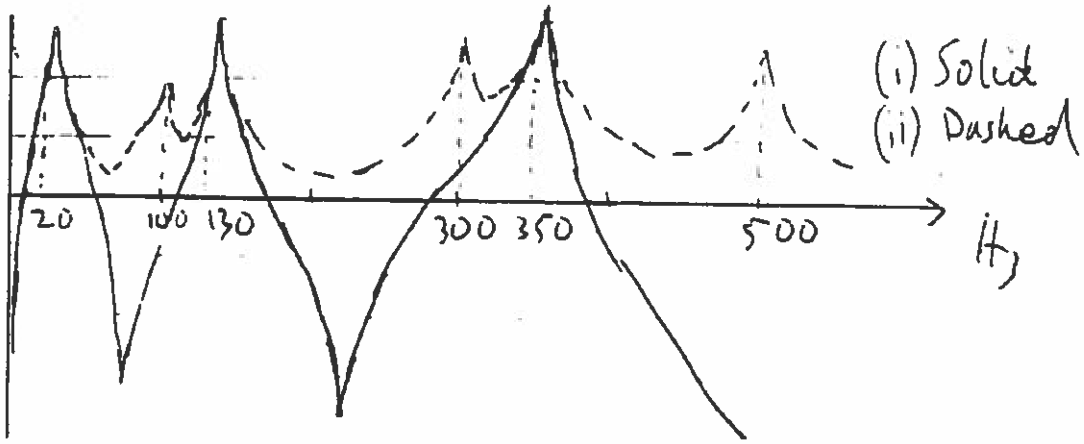
$\therefore$ $\underline{f_{sample} = 10000\ Hz}$ [20%]

(d) (i) (ii)



[20%]

(e)



Sign of $u_n(A)\,u_n(C)$ reverses for every mode pair: positive for bending, negative for torsion [20%]

Q1: Experimental modal analysis

A very popular question. Most candidates gave good explanations in (a), though some were confused about the bandwidth of a hammer pulse. Most gave a good answer to (b) but often missed allowing a saturation margin for the voltage output (target output of 4V or 4.5V rather than 5V). Part (c) was surprisingly poorly answered: many applied additional factors of two with spurious arguments; others provided sound logic but didn't allow a margin for the slow frequency roll-off beyond the 'useful' range. Sketches of flexural mode shapes in (d) were generally fine but would have benefitted from a bit more attention to the sequence of node points and boundary conditions; torsion mode shape sketches were not so good, with candidates not applying correct boundary conditions. Part (e) was well-answered by most; but a significant number incorrectly thought that flexural modes would not be visible in either of the transfer functions.

2(a) Because of the narrow neck, flow in the neck region carries most of the kinetic energy: flow inside the cavity is slow, and geometric spreading means it soon gets slow outside as well. Flow in the neck must be fairly uniform, so represent that kinetic energy via a mass moving at the mean flow speed.

(b) Change in internal volume $\delta V = AX + SY$

$\therefore$ change in density is $\delta \rho = -\rho \frac{\delta V}{V} = -\frac{\rho}{V}(AX+SY)$

So pressure change $\delta P = -\frac{c^2\rho}{V}(AX+SY) = -\frac{c^2\rho}{V}(x-y)$

with x, y as given.

Force on cone is $A\,\delta P$, force on piston is $S\,\delta P$

So equation of motion
$$\begin{cases} M\ddot{X} = -KX - \frac{c^2\rho}{V}(x-y)A \\ m\ddot{Y} = -\frac{c^2\rho}{V}(x-y)S \qquad (m=\rho SL) \end{cases}$$

i.e. $\frac{M}{A^2}\ddot{x} = -\frac{K}{A^2}x - \frac{c^2\rho}{V}(x-y)$, $\frac{m}{S^2}\ddot{y} = \frac{c^2\rho}{V}(x-y)$

These are the same as the governing equation for Fig 3, with the identification $k_a = \frac{K}{A^2}$, $k_b = \frac{c^2\rho}{V}$,

$m_a = \frac{M}{A^2}$, $m_b = \rho\frac{SL}{S^2} = \frac{\rho L}{S}$, so that

$$\begin{cases} m_a\ddot{x} = -k_a x - k_b(x-y) \\ m_b\ddot{y} = k_b(x-y) \end{cases}$$

(c)(i) Vent blocked $\rightarrow Y=0 \rightarrow y=0$

Then $m_a\ddot{x} = -(k_a+k_b)x$

$$\rightarrow \omega_{(i)}^2 = \frac{k_a+k_b}{m_a} = \frac{K+c^2A^2\rho/V}{M}$$

(ii) $x=0 \rightarrow m_b\ddot{y} = -k_b y$

$$\text{so } \omega_{(ii)}^2 = \frac{k_b}{m_b} = \frac{c^2\rho/V}{\rho L/S} = \frac{c^2 S}{LV},$$

same as Helmholtz resonator from Data Sheet.

2 (c)(iii) $K=0 \rightarrow k_a = 0$

One mode is rigid motion $x=y$, with $\omega = 0$

The second mode must be orthogonal to this, so

$$m_a x + m_b y = 0 \rightarrow y = -\frac{m_a}{m_b} x$$

Then $m_a \ddot{x} = -k_b\left(x + \frac{m_a}{m_b} x\right)$

so $$\omega^2 = \frac{k_b(1 + m_a/m_b)}{m_a} = k_b\left(\frac{1}{m_a} + \frac{1}{m_b}\right)$$

Interlacing: cases (i) and (ii) must interlace the general case, since a single constraint has been added

Case (iii) must give lower frequencies for both modes than the general case because only stiffness has been reduced. But no obvious interlacing connection.

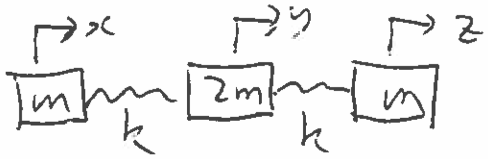
Case (ii) must interlace with case (iii), because if the cone is immobilised the value of K no longer matters.

(d) A bass reflex speaker has an extra resonance which can be used to extend the bass response for a given size of cabinet. Interlacing gives a clue: the vented case is bound to have a resonance below the single resonance of case (c)(i). However, it would not help to have case (c)(iii), in which case the extra "resonance" is right down at zero Hz (the rigid body mode). So careful design is needed to find the best compromise. In any case, the rigid mode of (c)(iii) has no net change of volume and so it is a very poor radiator of sound, no use for a loudspeaker.

Q2: Bass reflex loudspeaker

Most gave a satisfactory description in (a). Very few obtained the correct equivalent parameters in (b) despite managing most steps towards the equations of motion for the original system. Many forgot to comment on interlacing in (c). Only a few students understood the role of the vent/opening in part (d).

3(a)



Mass matrix $M = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$

Stiffness matrix $K = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$

System is symmetric, so one mode is symmetric $[1, 0, -1] = \underline{u}$

Then $K\underline{u} = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}$, $M\underline{u} = \begin{bmatrix} m \\ 0 \\ -m \end{bmatrix}$, so it is indeed a mode with frequency $\omega = \sqrt{k/m}$

One mode is rigid displacement $[1, 1, 1]$, $\omega = 0$

The third mode must be antisymmetric, $\underline{u} = [a, b, a]$ and orthogonal to the rigid mode, so $m(a + 2b + a) = 0$, ie $b = -a$

So try $\underline{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$: $K\underline{u} = \begin{bmatrix} 2k \\ -4k \\ 2k \end{bmatrix}$, $M\underline{u} = \begin{bmatrix} m \\ -2m \\ m \end{bmatrix}$, so $\omega = \sqrt{\frac{2k}{m}}$

(b) Now add the damper. Dissipation rate is $c(\dot{x} - \dot{y})^2$ so dissipation matrix is

$$C = \begin{bmatrix} c & -c & 0 \\ -c & c & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So from data sheet, $A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$

But $M^{-1} = \begin{bmatrix} 1/m & 0 & 0 \\ 0 & 1/2m & 0 \\ 0 & 0 & 1/m \end{bmatrix}$ so $M^{-1}K = \begin{bmatrix} k/m & -k/m & 0 \\ -k/2m & k/m & -k/2m \\ 0 & -k/m & k/m \end{bmatrix}$

so $$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k/m & k/m & 0 & -c/m & c/m & 0 \\ k/2m & -k/m & k/2m & c/2m & -c/2m & 0 \\ 0 & k/m & -k/m & 0 & 0 & 0 \end{bmatrix}$$

3 (b) cont. Each column of V gives a vector, consisting of $[x \; y \; z \; \dot{x} \; \dot{y} \; \dot{z}]^T$

$$= [x \; y \; z \; \lambda x \; \lambda y \; \lambda z]^T$$

where λ is the eigenvalue given in matrix D.

First 2 columns give a complex conjugate pair of modes with frequency $\mathrm{Im}(\lambda) = 44.57$.
Easiest to see the shape from the bottom half:

$$[0.58 \;,\; -0.57 + 0.07i \;,\; 0.56 - 0.02i]^t$$

This obviously matches the undamped mode $[1 \; -1 \; 1]^T$ so the undamped frequency was $\sqrt{2k/m} = \sqrt{2000} = 44.72$

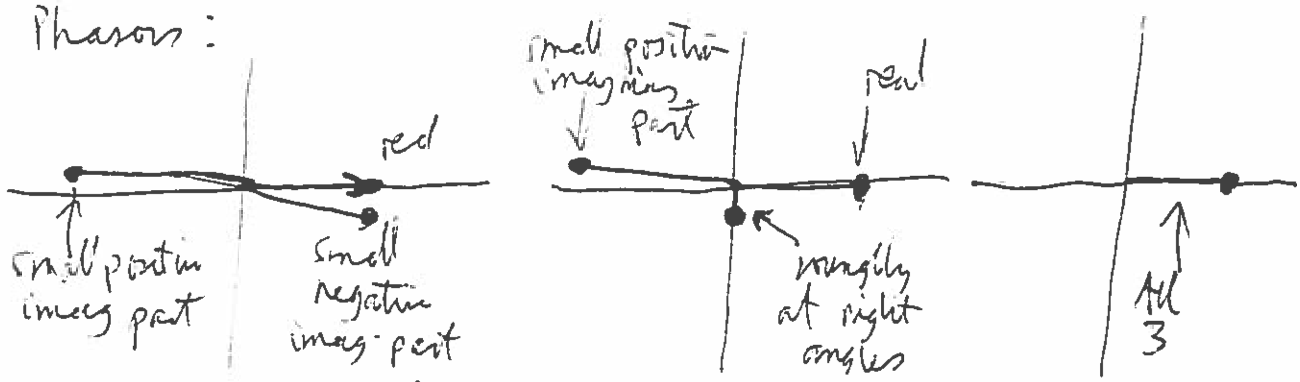
Second pair of columns gives shape

$$[-0.70 + .07i \;,\; -.00 - .03i \;,\; 0.71]^t$$

at frequency 31.69.
This matches undamped mode $[-1, 0, 1]^t$ at frequency $\sqrt{k/m} = \sqrt{1000} = 31.62$

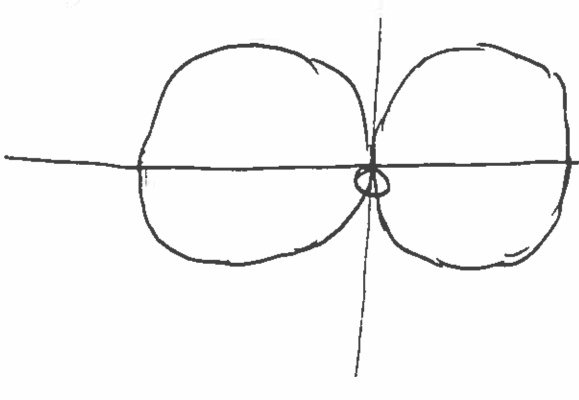
Final pair are the rigid-body mode with constant amplitude, zero frequency and no damping

(c) Phasors:



Intermediate mode is mode 2.

Circles:



Large circles to left and right, plus a small one rotated 90°

Q3: Mass-spring system

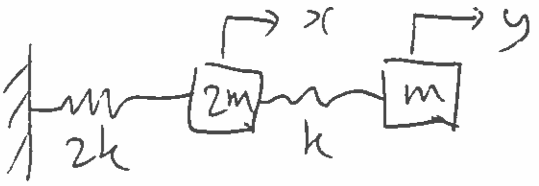
Part (a) was generally well answered but with several common slips: some did not account for the double mass in the middle when using orthogonality to calculate the third mode; some missed the rigid body mode; and others suggested spurious mode shapes. Many students did not attempt to calculate the matrix A, and few gave a clear explanation of how the frequencies and mode shapes related to the undamped system. Many students managed the phasor diagrams, though some interpreted the question to mean phasors of the complex frequencies rather than of the mode shapes. Only one student gave a good sketch of the three modal circles accounting for their phase and magnitude.

4(a)(i) The aluminium has very little intrinsic damping, and the sandwich panel will mainly be damped via the constrained layer effect. If the two plates were not glued together, simultaneous bending of the two would produce shear slippage at the interface. Putting a viscoelastic glue layer at that interface will result in cyclical shear strain in that layer, dissipating energy. There will also be some tensile/compressive strain cycling. The effect could be investigated quantitatively by the 3-layer method from the lectures.

(ii) When a plate vibrates with a magnet stuck to it, there are several routes for energy dissipation. There may be some shear slippage at the interface, especially near the corners where there is a stress concentration. Micro-slip is inevitably dissipative. If the motion is sufficiently vigorous that buzzing or rattling can be heard, other mechanisms come into play. It suggests that there is a "crack" between the magnet and the plate which is opening and closing. The buzzing noise means that impacts are occurring, transferring energy into different frequency ranges in a nonlinear manner, and carrying away energy. There will also be ~~aitt~~ air pumping in and out of the "crack", dissipating energy by viscosity.

(iii) A jet engine is a very hostile environment: very high temperature, high amplitude of noise and vibration, and potentially the presence of corrosive chemical agents. So it is not possible to have damping reliant on viscoelastic materials, which cannot stand these conditions. Only suitable metals have the required robustness. A friction damper makes use of gross sliding between components, which is a robust mechanism. Anything relying on air pumping or subtle effects from joints is likely to be unreliable given that it has to continue operating under a wide range of temperatures and operating levels.

4(B) (i)



Mass matrix $M = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}$

Stiffness matrix $K = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix}$

Natural frequencies: $\det \begin{bmatrix} 3k - 2m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} = 0$

$\therefore \ (3-2\lambda)(1-\lambda) - 1 = 0 \quad , \ \lambda = \omega^2 \frac{m}{k}$

$\therefore \ 2\lambda^2 - 5\lambda + 2 = 0$

$\therefore \ \lambda = \frac{1}{4}\left(5 \pm \sqrt{25-16}\right) = \frac{1}{2}, \ 2$

So $\omega^2 = \frac{k}{2m}, \ \frac{2k}{m}$

Mode shapes satisfy $3kx - ky = 2m\omega^2 x$

$\therefore \ y = \left(3 - 2\frac{m}{k}\omega^2\right)x = (3-2\lambda)x = 2x, -x$

So $\omega = \sqrt{\frac{k}{2m}}$ with vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\omega = \sqrt{\frac{2k}{m}}$ with $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(ii) New potential energy $V = \frac{1}{2} \cdot 2k x^2 + \frac{1}{2}k(1+i\eta)(x-y)^2$

Kinetic energy $T = \frac{1}{2} \cdot 2m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$

For a lightly-damped mode, use undamped mode in Rayleigh quotient to estimate modal damping.

$$\frac{1}{Q} = \frac{Im(\omega^2)}{Re(\omega^2)} \simeq \frac{\frac{1}{2}k\eta(x-y)^2}{\omega^2 (mx^2 + \frac{1}{2}my^2)}$$

So for mode 1, $Q_1 \simeq \dfrac{(k/2m)(m \cdot 1 + \frac{1}{2}m \cdot 4)}{\frac{1}{2}k\eta(1-2)^2} = \dfrac{3}{\eta}$

For mode 2, $Q_2 \simeq \dfrac{(2k/m)(m \cdot 1 + \frac{1}{2}m \cdot 1)}{\frac{1}{2}k\eta(1+1)^2} = \dfrac{3}{2\eta}$

Q4: Damping

A highly popular question. Part (a) was generally well-answered, though some did not read the question carefully in (iii) which was looking for a discussion of suitability rather than damping mechanisms. Most students made a good attempt at (b) (i) but many were let down by algebraic slips. One common error was to make incorrect guesses for the mode shapes sometimes on the basis of symmetry arguments, despite a lack of symmetry for this problem. Part (b) (ii) caused more problems and only a few correctly evaluated the Q factors, though many demonstrated that they knew the general approach.