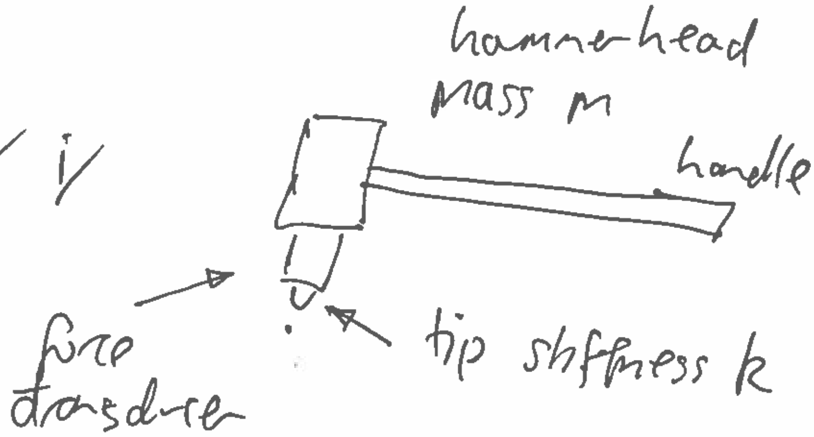
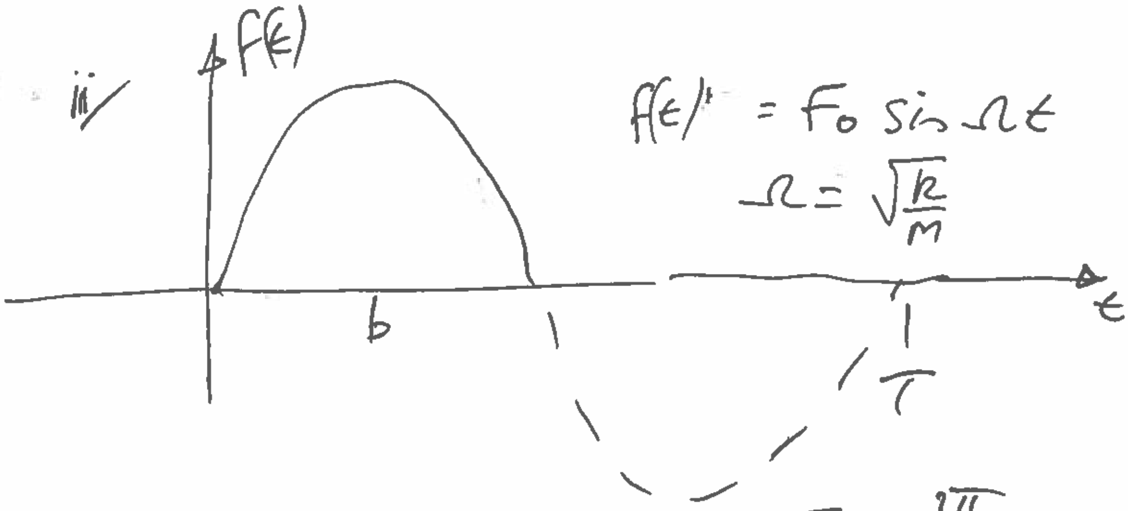
4C6 2009

Q1

a/ i/



ii/



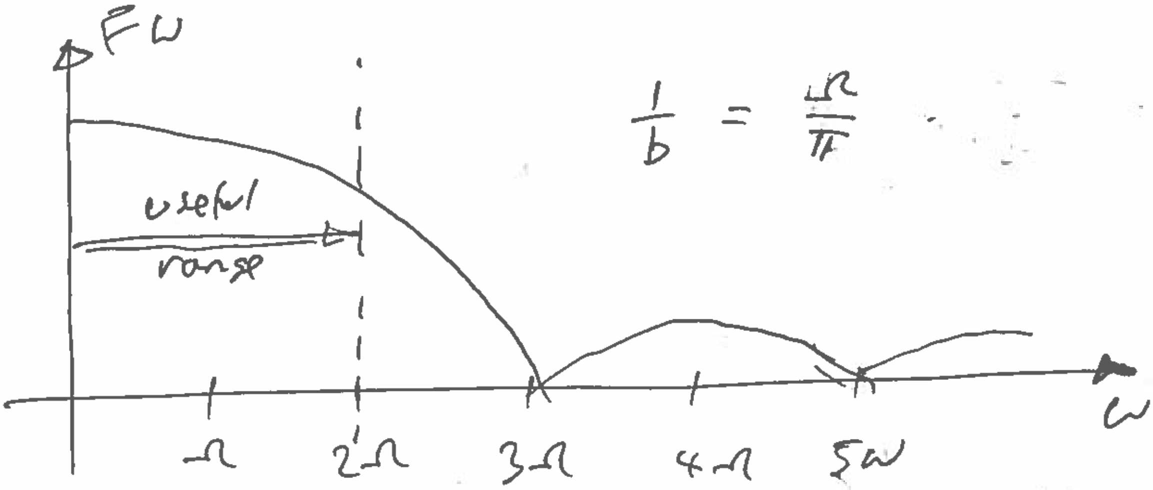
$$f(t) = F_0 \sin \Omega t$$

$$\Omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\Omega}$$

$$b = \frac{T}{2} = \frac{\pi}{\Omega} = \pi\sqrt{\frac{m}{k}}$$

iii/

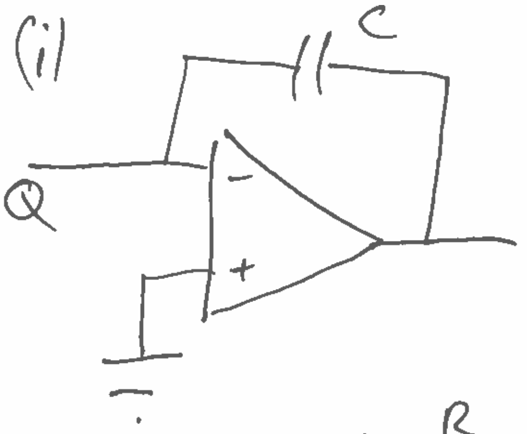


$$\frac{1}{b} = \frac{\Omega}{\pi}$$

useful range up to $\omega = 2\Omega = \frac{2\pi}{b}$ (rad/s)

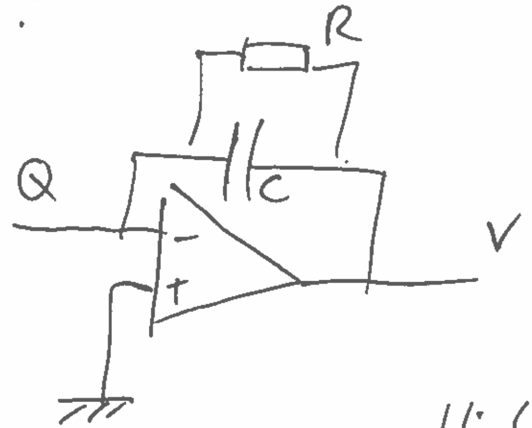
$$f = \frac{\omega}{2\pi} = \frac{1}{b} \quad (\text{Hz})$$

(b) (i)



$V = -\frac{Q}{C}$

ii)

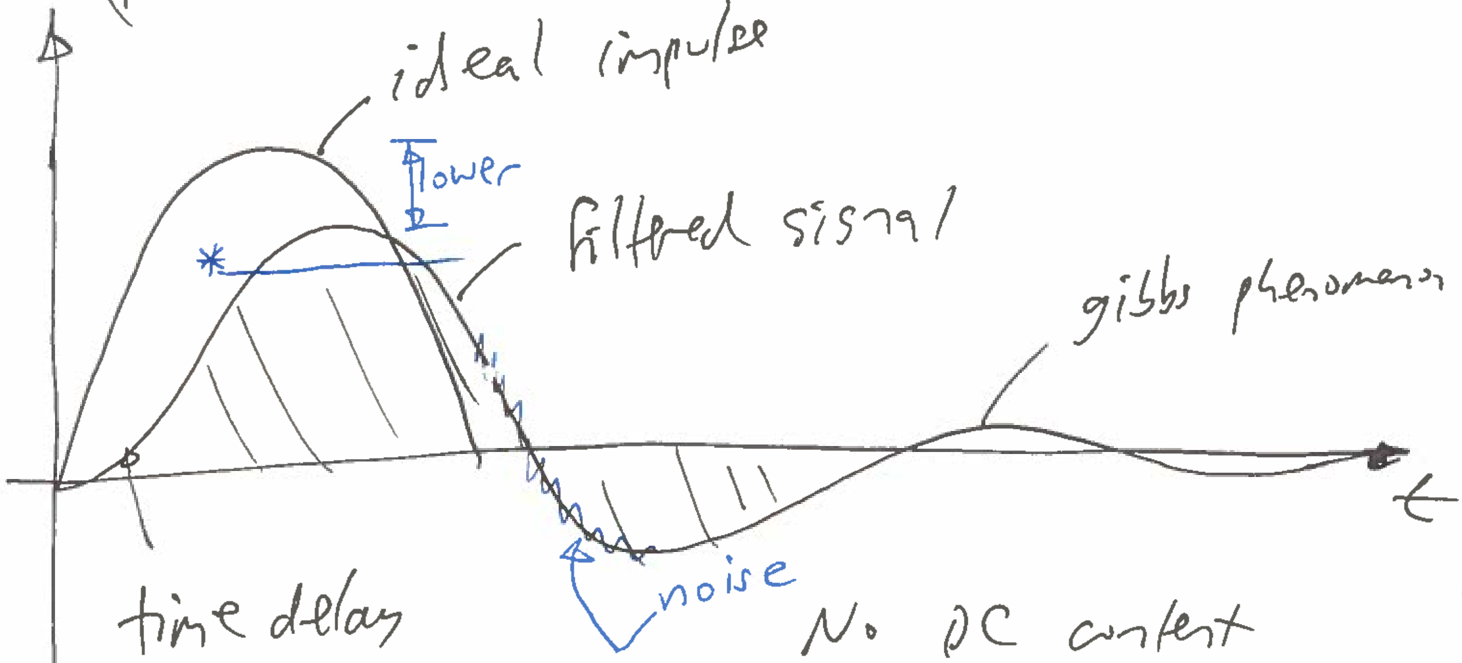


R prevents buildup of charge on C

High pass filter at frequency cutoff $\omega_c = \frac{1}{RC}$

iii Data logging, consider Nyquist criterion that ~~and~~ the sample rate is greater than twice the maximum frequency in the data. Suppose $f_{sample} = f_s$ then it makes sense to filter the signal with a low pass filter set below $\frac{f_s}{2}$. But if a soft hammer tip is used so that $\frac{1}{\tau} < \frac{f_s}{2}$ then no anti-aliasing filter is needed

(iv)

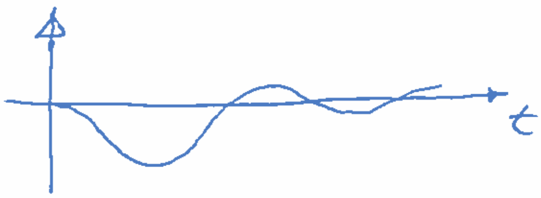


time delay due to low pass filter

No DC content due to high-pass filter (shaded areas are equal)

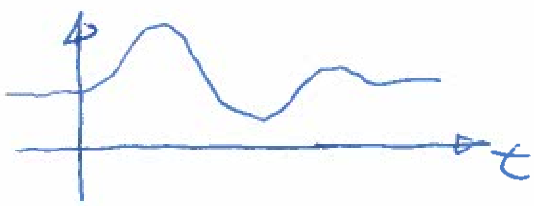
*clipping

inverted

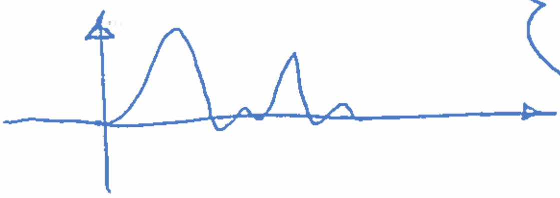


Since $V = -\frac{Q}{C}$

non-zero at $t=0$ due to a dc offset, or not settled down



double bounce



lots of things! Any three for 3/15 (= 20%) marks

4C6 Cribs 2019

2(a) $$T_1 \frac{\partial^2 w}{\partial x_1^2} + T_2 \frac{\partial^2 w}{\partial x_2^2} - m \frac{\partial^2 w}{\partial t^2} = 0$$

Separable solution, assume $w(x_1, x_2, t) = X_1(x_1) X_2(x_2) T(t)$

$$\Rightarrow \quad T_1 X_1'' X_2 T + T_2 X_2'' X_1 T - m X_1 X_2 T'' = 0$$

Divide through by $X_1 X_2 T$ to give:

$$T_1 \underbrace{\left(\frac{X_1''}{X_1}\right)}_{\substack{\text{Function of} \\ x_1 \text{ only} \\ \text{Say } -k_1^2 T_1 \\ (1)}} + T_2 \underbrace{\left(\frac{X_2''}{X_2}\right)}_{\substack{\text{Function of} \\ x_2 \text{ only} \\ \text{Say } -k_2^2 T_2 \\ (2)}} - m \underbrace{\left(\frac{T''}{T}\right)}_{\substack{\text{Function} \\ \text{of } t \text{ only} \\ \text{Say } -\omega^2 m \\ (3)}} = 0$$

← each term must be equal to a constant

(1) $\Rightarrow \quad X_1'' = -k_1^2 X_1 \Rightarrow X_1 = A \sin k_1 x_1 + B \cos k_1 x_1$

$X_1(0) = 0 \Rightarrow B = 0$

$X_1(L_1) = 0 \Rightarrow \sin k_1 L_1 = 0 \Rightarrow k_1 = \frac{n\pi}{L_1}$

(2) $\Rightarrow \quad X_2'' = -k_2^2 X_2 \Rightarrow X_2 = A \sin k_2 x_2 + B \cos k_2 x_2$

$\Rightarrow X_2(0) = 0 \Rightarrow B = 0$

$\Rightarrow X_2(L_2) = 0 \Rightarrow \sin k_2 L_2 = 0 \Rightarrow k_2 = \frac{m\pi}{L_2}$

(3) $\Rightarrow \quad T = A \cos \omega t + B \sin \omega t \Rightarrow$ SHM, as expected

so $W = \sin k_1 x_1 \sin k_2 x_2$ with $k_1 = \frac{n\pi}{L_1}$ and $k_2 = \frac{m\pi}{L_2}$ [30%]

(b) $m\omega^2 = T_1 k_1^2 + T_2 k_2^2$

$\Rightarrow$ $\underline{\omega_{nm}^2 = \left(\frac{T_1}{m}\right)\left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{T_2}{m}\right)\left(\frac{m\pi}{L_2}\right)^2}$ $n,m = 1,2,3..$ [20%]

(c) For different boundary conditions

$X_1 = A\sin k_1 x_1 + B\cos k_1 x_1$

$X_1'(0) = 0 \Rightarrow A = 0$

$X_1'(L_1) = 0 \Rightarrow \sin k_1 L_1 = 0$ as before $\Rightarrow k_1 = \frac{n\pi}{L_1}$

<u>But</u> in this case $k_1 = 0$ is also a possible solution

Similarly $X_2 = \cos k_2 x_2$ with $k_2 = \frac{m\pi}{L_2}$

$\Rightarrow$ $W = \cos k_1 x_1 \cos k_2 x_2$ $k_1 = \frac{n\pi}{L_1}$ $k_2 = \frac{m\pi}{L_2}$

$\underline{\omega_{nm}^2 = \left(\frac{T_1}{m}\right)\left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{T_2}{m}\right)\left(\frac{m\pi}{L_2}\right)^2}$ $n,m = 0,1,2,3$ [40%]

$\uparrow$

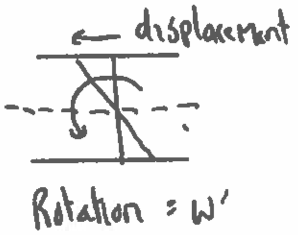
NB//

(d) Yes - the free membrane has additional modes (notes) [10%]

also location of nodal points is different so if struck at a given place the modal content will change

3 (a) this can be done from geometry of beam theory!

From geometry



Strain $= \frac{d}{dx}(dw')$

$\underline{\varepsilon = dw''}$

From beam theory $\sigma = Md/I$, $M = EIw'' \Rightarrow \sigma = Edw'' = E\varepsilon$

$\Rightarrow \underline{\varepsilon = dw''}$

Now strain energy $\underline{\text{density}}$ = $\frac{1}{2}\sigma\varepsilon = \frac{1}{2}E_a\varepsilon^2$

For total strain energy we need to integrate over the volume of the layer

$$\underline{U_a = \tfrac{1}{2}E_a t b d^2 \int_\alpha^\beta (w'')^2\,dx}$$

[25%]

(b) Note initially that $\int_\alpha^\beta w''^2\,dx = \int_\alpha^\beta \left(\frac{\pi}{L}\right)^4 \underbrace{\sin^2\frac{\pi x}{L}}_{\frac{1}{2}\left[1-\cos\frac{2\pi x}{L}\right]}\,dx$

$$= \left(\frac{\pi}{L}\right)^4 \frac{1}{2}\left[\beta-\alpha + \frac{L}{2\pi}\left(\sin\frac{2\pi\alpha}{L} - \sin\frac{2\pi\beta}{L}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{L}\right)^4 \gamma \quad \text{say}$$

Rayleigh Quotient $\omega_n^2 = \frac{U}{T'}$

$$\omega_n^2 = \frac{1}{T'}\left\{ \frac{1}{2}EI\underbrace{\int_0^L w''^2\,dx}_{\left(\frac{\pi}{L}\right)^4 \times \frac{1}{2}L} + \frac{1}{2}\underbrace{E_a}_{E_a(1+i\eta_a)} t b d^2 \times \frac{1}{2}\left(\frac{\pi}{L}\right)^4 \gamma \right\}$$

$$\omega_n^2 = \frac{1}{T'}\left\{ \tfrac{1}{4}\left(\frac{\pi}{L}\right)^4 EIL + \tfrac{1}{4}\left(\frac{\pi}{L}\right)^4 t b d^2 \gamma E_a (1+i\eta_a) \right\}$$

Now $\eta = \mathrm{Im}(\omega_n^2)/\mathrm{Re}(\omega_n^2)$

$$\Rightarrow \quad \eta = \frac{\eta_a E_a t b d^2 (\gamma/L)}{EI + E_a t b d^2 (\gamma/L)}$$

For maximum damping put treatment where w'' is greatest → centre of beam [40%]

(c) To put this result into the form of the data sheet, consider the denominator of the above expression

$$EI + E_a t b d^2 (\gamma/L) = EI\left[1 + \frac{E_a t b d}{EI}\right]$$

noting that $\gamma = L$ for full damping coverage

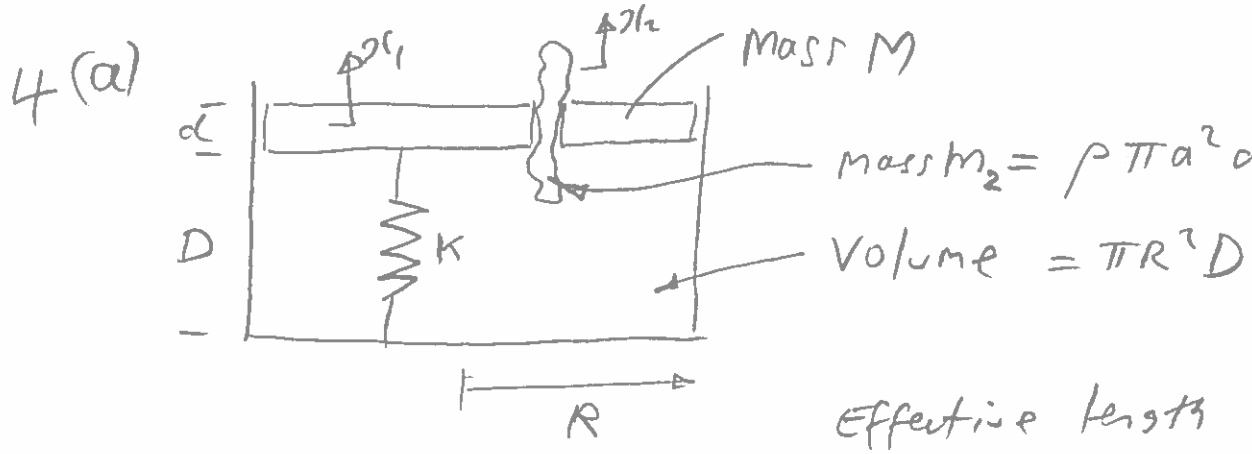
$$= E_1 I_1 \left[1 + \frac{E_2 h_2 b (h_1/2)^2}{E_1 b h_1^3/12} \right]$$

$$= E_1 I_1 [1 + 3eh] \quad \text{where } e = \frac{E_2}{E_1}; \; h = \frac{h_2}{h_1}$$

The data sheet has $E_1 I_1 [1 + eh^3 + 3(1+h)^2 eh/(1+eh)]$

Now if $h_1 \ll h_2$ then $h \ll 1$ and the two expressions agree [35%]

4(a)



Effective length d'
for "flanged circular neck"

$$d' = d + 1.7a$$

Increase in volume caused by x_1 and x_2
(assuming $R \gg a$) $\quad \Delta V = \pi R^2 x_1 - \pi a^2 x_2$

Change in density $= -\rho \frac{\Delta V}{V} = \frac{-\rho}{\pi R^2 D}\left(\pi R^2 x_1 - \pi a^2 x_2\right)$

but given $\Delta p = c^2 \Delta\rho \quad = \frac{-c^2\rho}{\pi R^2 D}\left(\pi R^2 x_1 - \pi a^2 x_2\right)$

EOM for piston

$$M\ddot{x}_1 = \Delta p \, \pi R^2 - K x_1$$

$$= -\left(\frac{c^2\rho\pi R^2}{D} + K\right)x_1 + \frac{c^2\rho\pi a^2}{D}x_2$$

EOM for hole

$$\underbrace{\left(\rho\pi a^2 d'\right)}_{m}\ddot{x}_2 = \Delta p \, \pi a^2$$

$$= -\frac{c^2\rho}{D}\frac{a^2}{R^2}\left(\pi R^2 x_1 - \pi a^2 x_2\right)$$

$$\therefore \begin{bmatrix} M & 0 \\ 0 & \rho\pi a^2 d' \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \frac{c^2\rho\pi R^2}{D}\begin{bmatrix} 1+K' & -\frac{a^2}{R^2} \\ -\frac{a^2}{R^2} & \frac{a^4}{R^4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $K' = \frac{KD}{c^2\rho\pi R^2}$

(b) $K' \to \infty \quad \therefore \quad x_1 = 0$

$$\therefore \rho\pi a^2 d' \ddot{x}_2 + \frac{c^2\rho\pi R^2}{D}\frac{a^4}{R^4} x_2 = 0$$

$$\therefore \omega_n^2 = \frac{c^2}{Dd'}\frac{a^2}{R^2}$$

[Compare with data sheet $\quad S = \pi a^2$, $L = d'$
$V = \pi R^2 D$

$$\omega^2 = c^2 \frac{S}{V''L''} = C^2 \frac{\pi a^2}{\pi R^2 D d'} = \frac{c^2}{Dd'}\frac{a^2}{R^2} \checkmark$$]

(c) $x_2 = 0 \quad$ (blocked hole)

$$\omega_n^2 = \frac{c^2\rho\pi R^2}{D}\frac{1+K'}{M}$$

$m_2 = \rho\pi a^2 d'$

(d) $K' = 0$

$A = \dfrac{c^2\rho\pi R^2}{D}$

$$\begin{vmatrix} A - M\omega^2 & -A\dfrac{a^2}{R^2} \\ -A\dfrac{a^2}{R^2} & A\dfrac{a^4}{R^4} - m_2\omega^2 \end{vmatrix} = 0$$

$$\therefore (A - M\omega^2)\left(A\frac{a^4}{R^4} - m_2\omega^2\right) - A^2\frac{a^4}{R^4} = 0$$

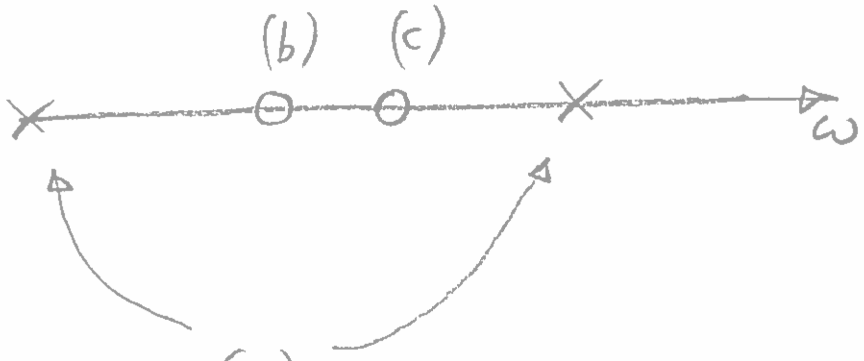
$$\therefore \omega^2\left[M m_2 \omega^2 - A\left(M\frac{a^4}{R^4} + m_2\right)\right] = 0$$

$\omega = 0 \quad$ rigid body mode

$$\omega^2 = \frac{c^2\rho\pi R^2}{D}\frac{M\dfrac{a^4}{R^4} + m_2}{M m_2} = \frac{c^2\rho\pi R^2}{D}\left(\frac{1}{m_2}\frac{a^4}{R^4} + \frac{1}{M}\right)$$

part (b) $\uparrow$ part $\uparrow$(c)

(e)



answers to (d)

answers to (b) & (c) fall between the two answers for (d) ∴ interlacing for application of constraint

(f) The casing at the bottom is shown thin so d' smaller ∴ m_2 smaller ∴ frequencies higher.

Many got lost in the algebra but quite a few got the answers out OK

4C6: Examiner's comments:

Q1 Impulse testing: 31 Attempts

Very popular and generally very well done, and most followed the methodology taught in lectures - easy marks, but showing that they had learned this useful stuff.

Q2 Rectangular membrane: 28 Attempts

Quite well done. Popular. In (a) "show that … must…" required a proof, not just a substitution. And care in explaining why n=0 is valid for (c) but not for (b). Lots of new modes so (d) is "yes" – most said "no".

Q3 Damping treatment 18 Attempts

Done very well by those who could follow instructions and answer the question. Many took it to be axial stretching of a bar. And many found the algebra overwhelming. The good answers where compact and clear.

Q4 Helmholtz 2dof: 25 Attempts

Many made a complete hash of this by not reading the question. It's a 2dof problem so it's easiest to set up a matrix equation.. The best answers again were compact and clear. Quite a few wrote pages and pages of rambly equations getting nowhere.