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RANDOM AND NON-LINEAR VIBRATION CRIBS 2015

1) a) $$m\ddot{x} + b\dot{x} + kx + \lambda v = ma \qquad \text{— (1)}$$

$$C\dot{v} + v/R = \lambda\dot{x} \qquad \text{— (2)}$$

Put $x(t) = x(\omega)e^{i\omega t}$ and $v(t) = v(\omega)e^{i\omega t}$, $a(t) = a(\omega)e^{i\omega t}$

$$\Rightarrow \left.\begin{aligned}(-m\omega^2 + i\omega b + k)x + \lambda v &= ma \\ (Ci\omega + 1/R)v &= \lambda i\omega x\end{aligned}\right\} \Rightarrow \underbrace{\left(\tfrac{1}{m}\right)\left[-m\omega^2 + i\omega b + k + \left(\frac{\lambda^2 i\omega}{Ci\omega + 1/R}\right)\right]}_{H^{-1}(\omega)} x = a$$

$$H^{-1}(\omega)x = a$$

So $x = H(\omega)a \Rightarrow \dot{x} = i\omega H(\omega)a \Rightarrow \underline{S_{\dot{x}\dot{x}} = |i\omega H|^2 S_{aa}(\omega)}$

For white noise input $\underline{S_{\dot{x}\dot{x}}(\omega) = \omega^2|H|^2 S_0}$ where H is defined above. [30%]

b) For $C=0$ equation (2) yields $v = \lambda R\dot{x}$

Equation (1) then becomes $m\ddot{x} + (b+\lambda^2R)\dot{x} + kx = ma$

Standard results for a system with white noise input $m\ddot{x} + c\dot{x} + kx = F$ $\quad S_{FF}(\omega) = S_0$

$$\Rightarrow \sigma_x^2 = \pi S_0/(ckm^2) \quad \times m^2$$

$$\sigma_{\dot{x}}^2 = \pi S_0/(cm^3) \quad \times m^2$$

Here $c = b+\lambda^2R$ and $F = ma \Rightarrow S_{FF} = m^2S_0$. Thus:

$$\underline{\sigma_x^2 = \frac{\pi S_0 m^2}{(b+\lambda^2R)k}}, \qquad \underline{\sigma_{\dot{x}}^2 = \frac{\pi S_0 m^2}{(b+\lambda^2R)m}} \qquad [30\%]$$

Power dissipated by damper $P_1 = b\dot{x}^2 \Rightarrow E[P_1] = b\sigma_{\dot{x}}^2$

by resistor $P_2 = \lambda^2R\dot{x}^2 \Rightarrow E[P_2] = \lambda^2R\sigma_{\dot{x}}^2$

$$\underline{E[P_1+P_2] = \frac{\pi S_0}{m}\times m^2} = \underline{\pi S_0 m}$$

c) $\nu_b^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) e^{-\frac{1}{2}(b/\sigma_x)^2}$

$P_{fail} = 1 - e^{-\nu_b^+ T}$

Number of peaks in 1 hour $= \nu_0^+ T_1 = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) T_1$ $\qquad T_1 = 60 \times 60\ s = 1$ hour

$= 1000$

$\Rightarrow$ When $T = 3 \times T_1$ then $\nu_0^+ T = 3000 \Rightarrow \nu_b^+ T = 3000 \times e^{-\frac{1}{2}(b/\sigma_x)^2}$

For $b = 5\sigma_x$ then $\nu_b^+ T = 3000\, e^{-\frac{1}{2}5^2} = 0.0112$

$\Rightarrow P_{Fail} = 1 - e^{-0.0112} = \underline{0.0111}$ [30%]

d) Instead of $S_{aa} = S_0$ we have $S_{aa} = \omega^2 S_0$

For $S_{\dot{x}\dot{x}}(\omega)$ numerator $\rightarrow \omega^k$ For large ω

denominator $\rightarrow \omega^k$

$\Rightarrow \int_{-\infty}^{\infty} S_{\dot{x}\dot{x}}(\omega)d\omega \rightarrow \infty$, non-physical [10%]

2) a) $p(x,\dot{x}) = C \exp\{-(\dot{x}/a)^2 - (x/d)^4\}$

$$\nu_b^+ = \int_0^\infty \dot{x}\, p(\dot{x}, b)\, d\dot{x} = C e^{-(x/d)^4} \underbrace{\int_0^\infty \dot{x} e^{-(\dot{x}/a)^2} d\dot{x}}_{\int_0^\infty -\frac{1}{2}a^2 \frac{d}{d\dot{x}}\left[e^{-(\dot{x}/a)^2}\right] d\dot{x} = \frac{1}{2}a^2} \quad \text{with } x = b$$

$$\Rightarrow \underline{\nu_b^+ = \tfrac{1}{2} C a^2 e^{-(b/d)^4}}$$

[25%]

b) Number of peaks above b in time T $= \nu_b^+ T$

Total number of peaks $= \nu_0^+ T$

Probability a peak is $> b$ $= \nu_b^+ / \nu_0^+ = 1 - P(b)$; $P(b)$ = probability Function

$$p(b) = \frac{d}{db} P(b) = -\frac{1}{\nu_0^+} \frac{d}{db}(\nu_b^+) = -\frac{d}{db} e^{-(b/d)^4}$$

$$\underline{p(b) = \frac{4b^3}{d^4} e^{-(b/d)^4}}$$

[25%]

c) Stress amplitude = $\beta \times$ peak hieght $= \beta b$.

$$\Rightarrow \quad 1/N = \alpha^{-1} S^2 = (\beta^2/\alpha)\, b^2 \Rightarrow E[1/N] = (\beta^2/\alpha)\, E[b^2]$$

$$E[b^2] = \int_0^\infty b^2 p(b)\, db = \frac{4}{d^4}\int_0^\infty b^5 e^{-(b/d)^4} db$$

Put $b^2 = y \Rightarrow db = dy/(2b) = \frac{1}{2} y^{-1/2} dy$

$b^5 = y^{5/2}$

Note - only two students correctly changed the variables on this integral

$$E[b^2] = \frac{2}{d^4} \underbrace{\int_0^\infty y^2 e^{-(y/d^2)^2} dy}$$

Standard integral. Put $1/d^4 = 1/2\sigma^2$, result is $\frac{1}{2}\sqrt{2\pi}\,\sigma^3 = \frac{1}{2}\sqrt{2\pi}\,\frac{1}{2\sqrt{2}}\,d^6$

$$E[b^2] = \frac{2}{d^4} \times \frac{\sqrt{\pi}}{4} d^6 = \frac{\sqrt{\pi}}{2} d^2$$

$$\Rightarrow \underline{E[1/N] = \left(\frac{\beta^2\sqrt{\pi}}{2\alpha}\right) d^2}$$

[25%]

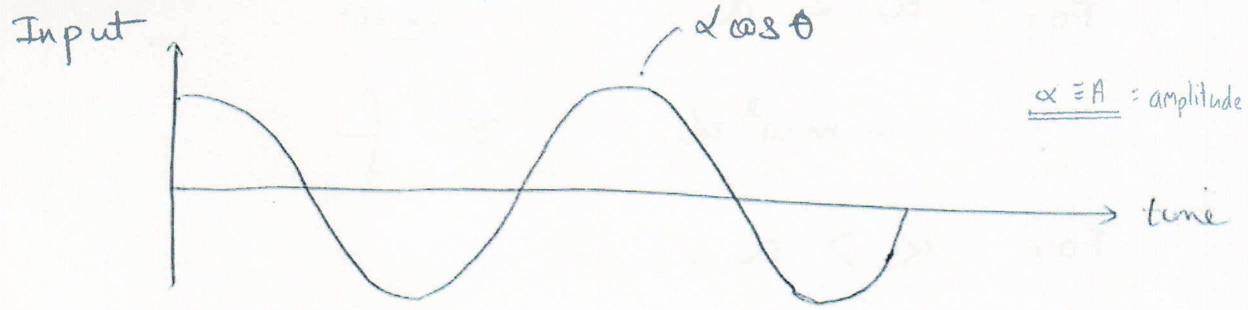
d) Number of peaks in time T $= \nu_0^+ T$

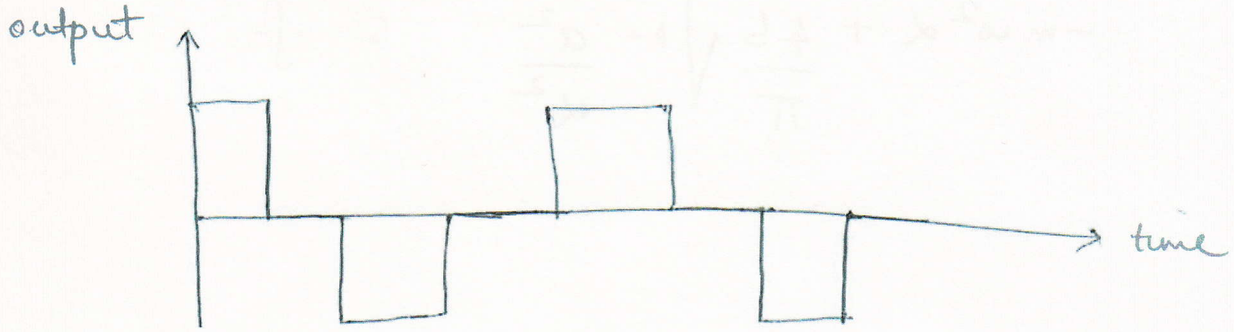
Fatigue damage in time T $= \nu_0^+ T \times E[1/N] = D$

For failure $D = 1 \Rightarrow T = \{\nu_0^+ E[1/N]\}^{-1}$

$E[1/N] = \left(\frac{\beta^2\sqrt{\pi}}{2\alpha}\right)d^2$; $\nu_0^+ = \frac{1}{2}Ca^2 \Rightarrow \underline{T = \frac{4\alpha}{\sqrt{\pi}\, C\, \beta^2 d^2 a^2}}$ [25%]

Q3 (a)





(b) Describing Function $= 0$ for $\alpha < a$

For the case $\alpha > a$

$$D.F. = \frac{1}{\alpha\pi}\int_0^{2\pi} (\text{output}) \cos\theta \, d\theta.$$

$$= \frac{4}{\alpha\pi}\int_0^{\beta} b \cos\theta \, d\theta \quad ; \beta = \cos^{-1}(a/\alpha).$$

$$= \frac{4b}{\alpha\pi} \sin\beta$$

$$= \frac{4b}{\alpha\pi}\sqrt{1 - \frac{a^2}{\alpha^2}}$$

(c) The equation of motion for the system in the Describing Function approximation is:

$$m\ddot{x} + (D.F) x \simeq f \cos\omega t.$$

For $\alpha < a$ $(\alpha \equiv 0)$

$$-m\omega^2 \alpha \simeq f$$

For $\alpha > a$

$$-m\omega^2 \alpha + \frac{4b}{\pi}\sqrt{1 - \frac{a^2}{\alpha^2}} \simeq f.$$

Q4

(a) Singular points given by:

$$\dot{x} = 0 \quad \text{and} \quad \dot{y} = 0$$

$$y - y^3 = 0 \Rightarrow y = 0, \; y = \pm 1$$

and $-\alpha x - y^2 = 0 \Rightarrow$ for $y = 0$, $x = 0$

for $y = \pm 1$, $-\alpha x - 1 = 0$

$\therefore$ for $y = \pm 1$, $x = -1/\alpha$

$\therefore$ singular points are $(0,0), (-1/\alpha, 1), (-1/\alpha, -1)$

(b) Linearising about $(0, 0)$

Note - the phase plane is (x,y). Some students considered $(y,\dot{y})$, which is not intended

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix}$$

eigenvalues given by:

$$\begin{vmatrix} -\lambda & 1 \\ -\alpha & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + \alpha = 0$$

$$\lambda = \pm\sqrt{(-\alpha)}$$

$\therefore$ centre if $\alpha > 0$ and saddle point if $\alpha < 0$

Linearising about $(-1/\alpha, 1)$

$$A = \begin{bmatrix} 0 & -2 \\ -\alpha & -2 \end{bmatrix}$$

eigenvalues given by:

$$\begin{vmatrix} -\lambda & -2 \\ -\alpha & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 2\lambda - 2\alpha = 0$$

$$\Rightarrow \lambda = -1 \pm \sqrt{1 + 2\alpha}$$

$\therefore$ saddle point if $\alpha > 0$, stable node for ~~or~~ $\alpha < 0$

and stable focus for $\alpha < -0.5$

Linearising about $(-1/\alpha, -1)$

$$A = \begin{bmatrix} 0 & -2 \\ -\alpha & 2 \end{bmatrix}$$

eigenvalues of A given by:

$$\begin{vmatrix} -\lambda & -2 \\ -\alpha & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda - 2\alpha = 0$$

$$\lambda = 1 \pm \sqrt{1+2\alpha}$$

$\therefore \alpha > 0 \Rightarrow$ saddle point, if $-0.5 < \alpha < 0 \Rightarrow$ unstable node and if $\alpha < -0.5 \Rightarrow$ unstable focus.

(c) origin is a centre and the other two equilibrium points are saddle points