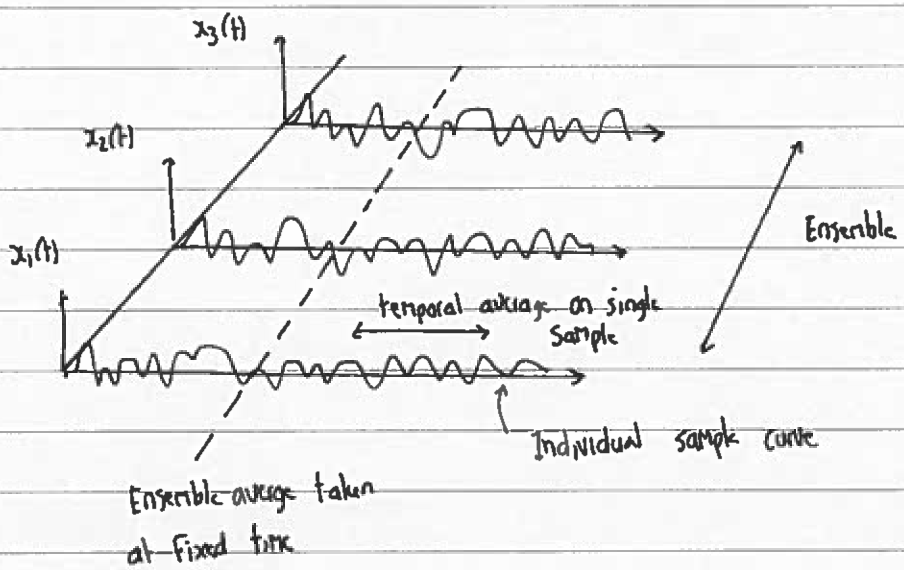
Prof R S Langley

RANDOM AND NONLINEAR VIBRATIONS - CRIBS 2016

1. a) Ergodic - time and ensemble averages are the same

Stationary - ensemble averages are independent of time



Ergodic ⇒ stationary : not possible for time and ensemble averages to differ if the process is ergodic.

Stationary ⇏ ergodic : For example an ensemble of sine curves with random amplitudes [20%]

b) (i) $\iint p(a,\dot{a})\,da\,d\dot{a} = 1$

$$\Rightarrow C\underbrace{\int_0^\infty a e^{-\frac{1}{2}(a/\alpha)^2}\,da}_{\substack{\alpha^2 \\ \text{(Data sheet)}}}\ \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\dot{a}/\beta)^2}\,d\dot{a}}_{\substack{2\times\sqrt{\frac{\pi}{2}}\,\beta \\ \text{(Data sheet)}}} = 1 \Rightarrow C\sqrt{2\pi}\,\alpha^2\beta = 1 \Rightarrow \underline{C = \frac{1}{\sqrt{2\pi}\,\alpha^2\beta}}$$ [15%]

(ii) $$p(a) = \int_{-\infty}^{\infty} p(a,\dot{a})\,d\dot{a} = Ca e^{-\frac{1}{2}(a/\alpha)^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\dot{a}/\beta)^2}\,d\dot{a} = \left(\frac{a}{\alpha^2}\right)e^{-\frac{1}{2}(a/\alpha)^2}$$

$\Rightarrow p(a) = \left(\frac{a}{\alpha^2}\right)e^{-\frac{1}{2}(a/\alpha)^2}$ This is a Rayleigh distribution [15%]

(iii) $$P = \int_0^b p(a)\,da = \left[-e^{-\frac{1}{2}(a/\alpha)^2}\right]_0^b = 1 - e^{-\frac{1}{2}(b/\alpha)^2}$$ [15%]

(iv) $\quad P = e^{-\nu_b^+ T}$ where $\nu_b^+ = \int_0^\infty \dot{a}\, p(a,\dot{a})\, d\dot{a}$ with $a \to b$

$$\Rightarrow \nu_b^+ = C a e^{-\frac{1}{2}(a/\alpha)^2} \underbrace{\int_0^\infty \dot{a}\, e^{-\frac{1}{2}(\dot{a}/\beta)^2} d\dot{a}}_{\beta^2 \text{ (data sheet)}} = \frac{1}{\sqrt{2\pi}}\left(\frac{\beta}{\alpha^2}\right) a e^{-\frac{1}{2}(a/\alpha)^2} \quad a \to b$$

$$\Rightarrow \underline{P = \exp\left\{-\frac{T}{\sqrt{2\pi}}\left(\frac{\beta}{\alpha^2}\right) b e^{-\frac{1}{2}(b/\alpha)^2}\right\}}$$ [35%]

2 (a) $\dot{F} + \alpha F = \alpha w(t)$

Assume $F = F(\omega) e^{i\omega t}$, $w = W(\omega) e^{i\omega t}$

$$\Rightarrow F(\omega) = \frac{\alpha W(\omega)}{i\omega + \alpha} \Rightarrow \text{F.R.F} = H_F(\omega) = \frac{\alpha}{i\omega + \alpha}$$

$$S_{FF}(\omega) = |H_F(\omega)|^2 S_0 = \underline{\left(\frac{\alpha^2}{\omega^2+\alpha^2}\right) S_0}$$

Now $\ddot{x} + 2\beta\omega_n \dot{x} + \omega_n^2 x = \gamma F$

$$\Rightarrow x(\omega) = \frac{\gamma F(\omega)}{-\omega^2 + \omega_n^2 + 2i\beta\omega\omega_n} \Rightarrow \text{F.R.F} = \frac{\gamma}{-\omega^2+\omega_n^2+2i\beta\omega\omega_n} = H_{x/F}(\omega)$$

So $S_{xx}(\omega) = |H_{x/F}(\omega)|^2 S_{FF}(\omega)$

$$\Rightarrow \underline{S_{xx}(\omega) = \left(\frac{\gamma^2\alpha^2}{(\omega_n^2-\omega^2)^2 + (2\beta\omega\omega_n)^2}\right)\left(\frac{1}{\omega^2+\alpha^2}\right) S_0}$$ [25%]

(b) $H_{x/F}$ strongly peaked at $\omega \approx \omega_n$; since $\alpha \gg \omega_n$ we can assume $\alpha \gg \omega$ so that:-

$$S_{xx}(\omega) \approx \frac{\gamma^2 S_0}{(\omega_n^2-\omega^2)^2 + (2\beta\omega\omega_n)^2}$$ - classic white noise SDOF result.

From data sheet $\underline{\sigma_x^2 = \frac{\pi\gamma^2 S_0}{2\beta\omega_n^3}}$ $\underline{\sigma_{\dot{x}}^2 = \frac{\pi\gamma^2 S_0}{2\beta\omega_n}}$ [25%]

(c) From data sheet $E[D] = E[1/N(S)]\, \nu_0^+ T$ ⟶ data sheet, Gaussian process

$\downarrow$

$\underline{E[S/\gamma] = E[\lambda b/\gamma]}$ $\nu_0^+ = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) = \frac{\omega_n}{2\pi}$

Now the distribution of b is given on data sheet $p(b) = \frac{b}{\sigma_x^2} e^{-\frac{1}{2}(b/\sigma_x)^2}$

$$E[\lambda b/\gamma] = (\lambda/\gamma)\int_0^\infty \frac{b^2}{\sigma_x^2} e^{-\frac{1}{2}(b/\sigma_x)^2}\, db = (\lambda/\gamma)\sqrt{\frac{\pi}{2}}\, \sigma_x$$ from data sheet.

Thus $$\underline{E[D] = \underbrace{(\lambda/\gamma)\sqrt{\frac{\pi}{2}}\,\sigma_x \times \left(\frac{\omega_n}{2\pi}\right)}_{\text{damage per unit time}} \times T}$$ [30%]

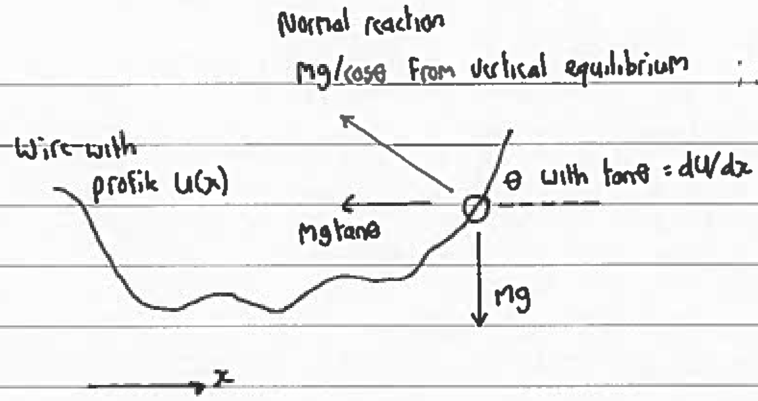
(d) Assumptions 1) Miner's law is valid

2) The process is narrow banded, so that the peaks are Rayleigh

3) The S-N law is accurate

Usually a factor of safety of 4 or 5 is applied. [20%]

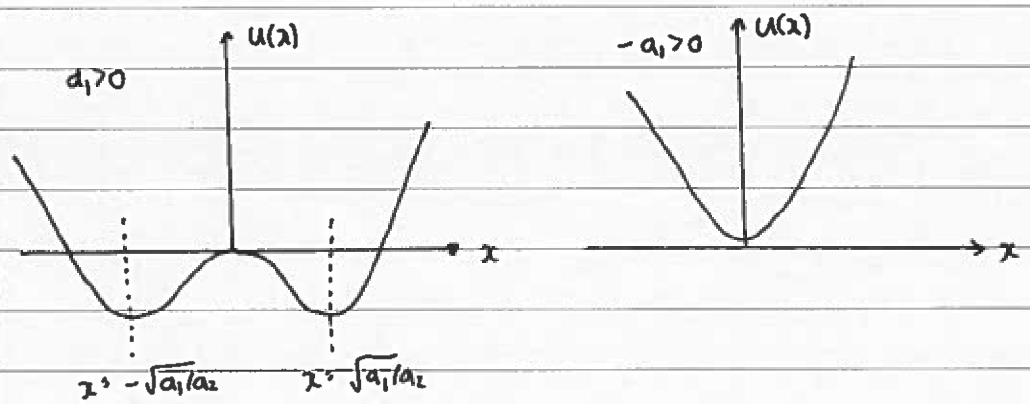
3 (a)



; small slope assumed, vertical inertia negligible.

Thus $m\ddot{x} = -mg\tan\theta \Rightarrow m\ddot{x} + mg\frac{dy}{dx} = 0 \Rightarrow \underline{\ddot{x} + g\frac{dU}{dx} = 0}$

Shape of wire has $g\frac{dU}{dx} = -a_1 x^3 + a_2 x^5 \Rightarrow \underline{gU = -\tfrac{1}{4}a_1 x^4 + \tfrac{1}{6}a_2 x^6}$



[20%]

(b) $\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ a_1 x^3 - a_2 x^5 \end{pmatrix}$ singular points at $\dot{x} = 0$ and $x = 0$

and $\dot{x} = 0$ and $x = \pm\sqrt{a_1/a_2}$ ← for $a_1 > 0$ only

Linearisation $x \to x_s + \delta x$ (x_s = singular point)

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x_s \\ \dot{x}_s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3a_1 x_s^2 - 5a_2 x_s^4 & 0 \end{pmatrix}\begin{pmatrix} x_s \\ \dot{x}_s \end{pmatrix}$$

Eigen problem $\left|\begin{pmatrix} -\lambda & 1 \\ 3a_1 x_s^2 - 5a_2 x_s^4 & -\lambda \end{pmatrix}\right| = 0$

$\underline{\text{For } x_s = 0}$ $\lambda^2 = 0$, two roots with $\lambda = 0$.

Indeterminate point, can determine nature from bead analogy.

For $a_1 < 0$, the situation is like two complex roots with zero real (and imaginary) parts.

$\Rightarrow$ centre

For $a_1 > 0$, the situation is like a positive and negative real root

$\Rightarrow$ saddle point

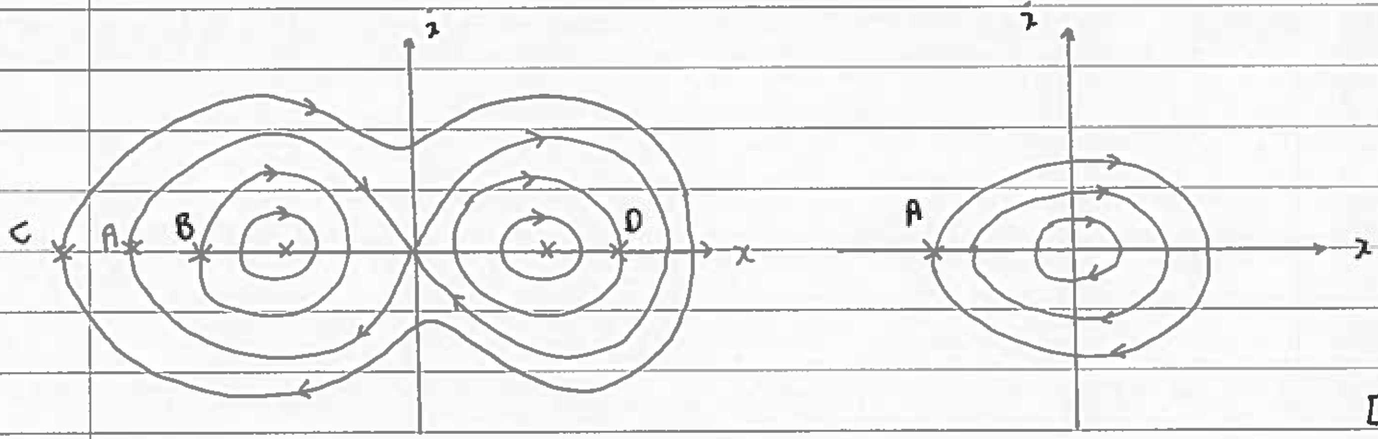
For $x_s = \pm\sqrt{a_1/a_2}$ ($a_1 > 0$ only) $\lambda^2 = 3a_1\left(\frac{a_1}{a_2}\right) - 5a_2\left(\frac{a_1}{a_2}\right)^2 = -2a_1^2/a_2$

$\Rightarrow \lambda = \pm i\sqrt{2a_1^2/a_2}$

$\Rightarrow$ centre

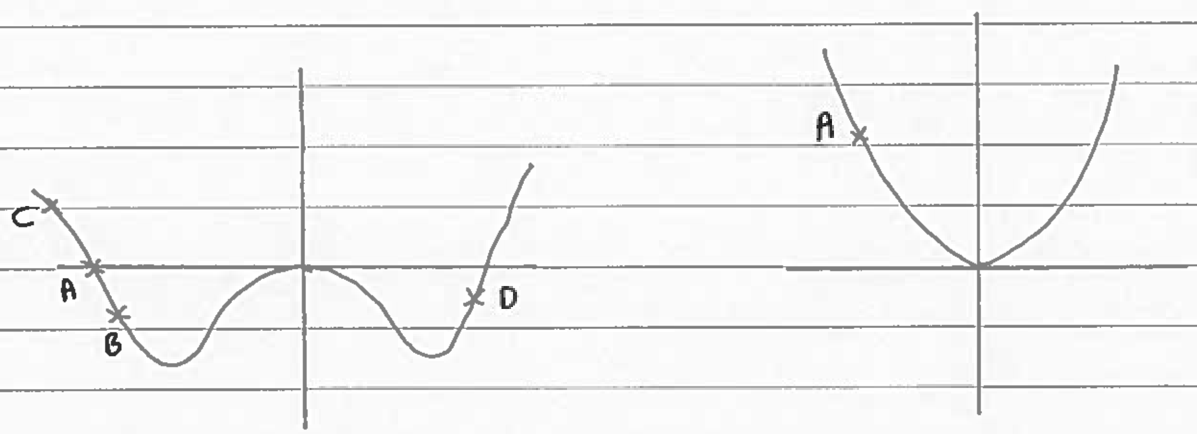
[40%]

(c) $a_1 > 0$ $a_1 < 0$



[20%]

(d) With reference to orbits labelled above



[20%]

4. (a) $\ddot{x} + b\dot{x} + cx|x| = B\cos\omega t$

$\downarrow$

Assume that $x = A\cos\omega t$ say.

In the D.F. approach we replace $cx|x|$ with γx such that:

$$\int_0^{2\pi/\omega} \gamma x^2\,dt = \int_0^{2\pi/\omega} c\,x^2|x|\,dt$$

$$\gamma A^2(\pi/\omega) \qquad\qquad 4A^3c\int_0^{\pi/2\omega}\cos^3\omega t\,dt = 4A^3c\int_0^{\pi/2\omega}\left(\tfrac{3}{4}\cos\omega t + \tfrac{1}{4}\cos 3\omega t\right)dt$$

$$= 4A^3c\left[+\frac{3}{4\omega}\sin\omega t + \frac{1}{12\omega}\sin 3\omega t\right]_0^{\pi/2\omega}$$

$$= \frac{4A^3c}{\omega}\left[\tfrac{3}{4} - \tfrac{1}{12}\right] = \frac{8}{3\omega}A^3c$$

Thus $\gamma A^2(\pi/\omega) = (8/3\omega)A^3c \Rightarrow \gamma = \left(\frac{8}{3\pi}\right)Ac$

$\Rightarrow$ Linearised equation is : $\underline{\ddot{x} + b\dot{x} + \left(\frac{8}{3\pi}\right)Acx = B\cos\omega t}$ [40%]

(b) $B = b = 0 \Rightarrow \ddot{x} + \left(\frac{8}{3\pi}\right)Acx = 0$

$x = A\cos\omega t \Rightarrow -\omega^2 A + \left(\frac{8}{3\pi}\right)cA^2 = 0$

$\Rightarrow \omega = \left[\left(\frac{8}{3\pi}\right)cA\right]^{1/2}$

$\Rightarrow \omega \propto A^{1/2}$, $\underline{T \propto A^{-1/2}}$ [10%]

(c) $b = 0 \Rightarrow \ddot{x} + \underbrace{\left(\frac{8}{3\pi}\right)c}_{\text{call this } \beta}Ax = B\cos\omega t$

$\ddot{x} + \beta Ax = B\cos\omega t$

$x = A\cos\omega t$ NB this is always positive, should be written as $|A|$

$-\omega^2 A + \beta A|A| = B$

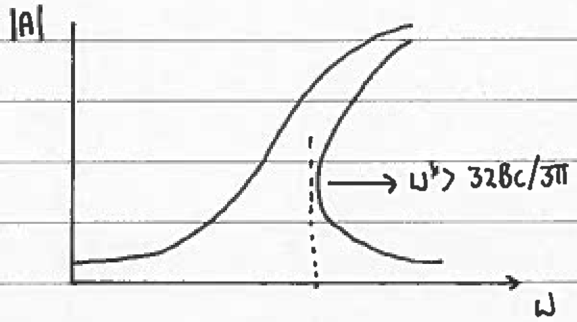
(i) Assume A is positive $|A| = A \Rightarrow -\omega^2 A + \beta A^2 = B \Rightarrow A = \dfrac{\omega^2 \pm \sqrt{\omega^4 + 4\beta B}}{2\beta}$

Only the +ve root is consistent with +ve $A \Rightarrow$ single root $\underline{A = \frac{1}{2\beta}\left\{\omega^2 + \sqrt{\omega^4 + 4\beta B}\right\}}$

(ii) Assume A is negative $|A| = -A$ $\Rightarrow$ $-\omega^2 A - \beta A^2 = B$ $\Rightarrow$ $A = \dfrac{-\omega^2 \pm \sqrt{\omega^4 - 4\beta B}}{2\beta}$

This solution only exists for $\omega^4 > 4\beta B \Rightarrow \underline{\omega^4 > (32B/3\pi)c}$

In this case there are two solutions $\Rightarrow$ 3 solutions in total



Depending on initial conditions, different solutions may be obtained (but only top and bottom solutions are stable). [30%]

(d) Assume $x = A e^{i\omega t}$ $\Rightarrow$ $\underline{\left\{-\omega^2 + i\omega b + \left(\tfrac{8}{3\pi}\right)c|A|\right\}A = B}$ [20%]

A now complex

need a numerical solution (typically) to find the real and imaginary parts of A