4C7 Random Vibration 2017

1 (a) $\quad \varepsilon = kx^3 - k_e x$

$$E[\varepsilon^2] = E[(kx^3 - k_e x)^2]$$

$$\frac{\partial}{\partial k_e} E[\varepsilon^2] = E[-2x(kx^3 - k_e x)] = 0$$

$$\Rightarrow \quad E[kx^4] = E[k_e x^2]$$

$$\Rightarrow \quad 3k\sigma_x^4 = k_e \sigma_x^2 \quad \text{(see "hint")}$$

$$\Rightarrow \quad \underline{k_e = 3k\sigma_x^2}$$

[20%]

(b) $\quad m\ddot{x} + c\dot{x} + k_1 x + k_3 x^3 = F$

Linearise $\Rightarrow \quad m\ddot{x} + c\dot{x} + (k_1 + k_e)x = F \qquad k_e = k_3\sigma_x^2 \times 3$

From data sheet, response to white noise : $\sigma_x^2 = \dfrac{\pi S_0}{c(k_1 + k_e)}$

$$\Rightarrow \quad \sigma_x^2(k_1 + 3k_3\sigma_x^2) = \frac{\pi S_0}{c}$$

$$\Rightarrow \quad 3k_3\sigma_x^4 + k_1\sigma_x^2 = \frac{\pi S_0}{c}$$

$$\Rightarrow \quad \underline{\sigma_x^2 = \frac{-k_1 + \sqrt{k_1^2 + 12k_3\pi S_0/c}}{6k_3}}$$

NB/ as a check, can be shown that $\sigma_x^2 \to \dfrac{\pi S_0}{ck_1}$ when $k_3 \to 0$ [30%]

(c) $\quad P = E[c\dot{x} \times \dot{x}] = c\sigma_{\dot{x}}^2$

From data sheet $\sigma_{\dot{x}}^2 = \dfrac{\pi S_0}{cm} \quad \Rightarrow \quad \underline{P = \dfrac{\pi S_0}{m}}$ [10%]

(d) Apply linearisation $\quad c\dot{x}^5 = c_e\dot{x} + \varepsilon$

$$\varepsilon = c\dot{x}^5 - c_e\dot{x}$$

$$\frac{\partial}{\partial c_e} E[\varepsilon^2] = E[-2\dot{x}(c\dot{x}^5 - c_e\dot{x})]$$

$$\Rightarrow \quad c_e = cE[\dot{x}^6]/E[\dot{x}^2] = c15\sigma_{\dot{x}}^4 \quad \text{(see "hint")}$$

Then from data sheet $\sigma_{\dot{x}}^2 = \dfrac{\pi S_0}{c_e m} \quad \Rightarrow \quad \underline{\sigma_{\dot{x}} = \left(\dfrac{\pi S_0}{15cm}\right)^{1/6}} \quad ; \quad P = c_e E[\sigma_{\dot{x}}^2] = \underline{\dfrac{\pi S_0}{m}}$ [40%]

2 (a) By inspection, the equations of motion are:

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

For frequency response function, assume all variables are proportional to $e^{i\omega t}$

$$\Rightarrow \begin{pmatrix} -\omega^2 M_1 + ci\omega + 2k & -k \\ -k & -\omega^2 M_2 + k \end{pmatrix}\begin{pmatrix} x_1(\omega) \\ x_2(\omega) \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1(\omega) \\ x_2(\omega) \end{pmatrix} = \frac{1}{\text{Det}}\begin{pmatrix} -\omega^2 M_2 + k & k \\ k & -\omega^2 M_1 + ci\omega + 2k \end{pmatrix}\begin{pmatrix} F \\ 0 \end{pmatrix}$$

$\text{Det} = (-\omega^2 M_1 + ci\omega + 2k)(-\omega^2 M_2 + k) - k^2$ [Further simplification not needed]

Thus $y(\omega) = x_2(\omega) - x_1(\omega) = \left(\frac{\omega^2 M_2}{\text{Det}}\right) F(\omega)$

$$\Rightarrow \underline{S_{yy}(\omega) = \left|\frac{\omega^2 M_2}{\text{Det}}\right|^2 S_{FF}(\omega)}$$

Zero for $M_2 = 0$ because there is no inertial loading on the end of the second spring, and so the spring has no extension $\Rightarrow x_1 = x_2$. [50%

(b) $y = x_2 - x_1 \Rightarrow E[y^2] = E[x_2^2] + E[x_1^2] - 2E[x_1 x_2]$

$= \sigma_x^2 + \sigma_x^2 - 2\rho\sigma_x^2 = 2\sigma_x^2(1-\rho)$ where $\sigma_x^2 = E[x_1^2] = E[x_2^2]$

Similarly $\dot{y} = \dot{x}_2 - \dot{x}_1 \Rightarrow E[\dot{y}^2] = 2\sigma_{\dot{x}}^2(1-\rho)$; $\sigma_{\dot{x}}^2 = E[\dot{x}_1^2] = E[\dot{x}_2^2]$

Mean rate of impact $= \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dot{y}}}{\sigma_y}\right) e^{-\frac{1}{2}\left(\frac{b}{\sigma_y}\right)^2}$ From data sheet $= \nu_b^+$

For $\rho = -0.5$ $\sigma_y^2 = 3\sigma_x^2 = 3$ mm

$\sigma_{\dot{y}}^2 = 3\sigma_{\dot{x}}^2 = 3 \times 6 \times 10^5$ mm^{2}s^{-2}

$\Rightarrow \nu_b^+ = 0.2695$ (very high) From data sheet $P_F = 1 - e^{-\nu_b^+ T} = 1 - e^{-\nu_b^+ \times 30} = \underline{0.9996}$

For $\rho = 0.5$ $\sigma_y^2 = \sigma_x^2 = 1$ mm, $\sigma_{\dot{y}}^2 = \sigma_{\dot{x}}^2 = 6 \times 10^5$ mm^{2}s^{-2}

$\Rightarrow \nu_b^+ = 1.533 \times 10^{-6}$; $P_F = 1 - e^{-1.533 \times 10^{-6} \times 30} = \underline{4.59 \times 10^{-5}}$

Gap goes from $\underline{b = 3.5\sigma_y}$ with negative correlation to $\underline{b = 6 \times \sigma_y}$ with +ve correlation $\Rightarrow$ large change in probability [50%

Q3 (a) System dynamics represented by:

$$\dot{x} = y$$
$$\dot{y} = x - x^2$$

Singular points when $\dot{x} = \dot{y} = 0$

$\dot{x} = 0, \; x = 0$ and $\dot{x} = 0, \; x = 1$

For singular point $(0,0)$ linearisation results in $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and eigenvalues given by

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \text{ or } \lambda = \pm 1$$

$\therefore$ saddle point

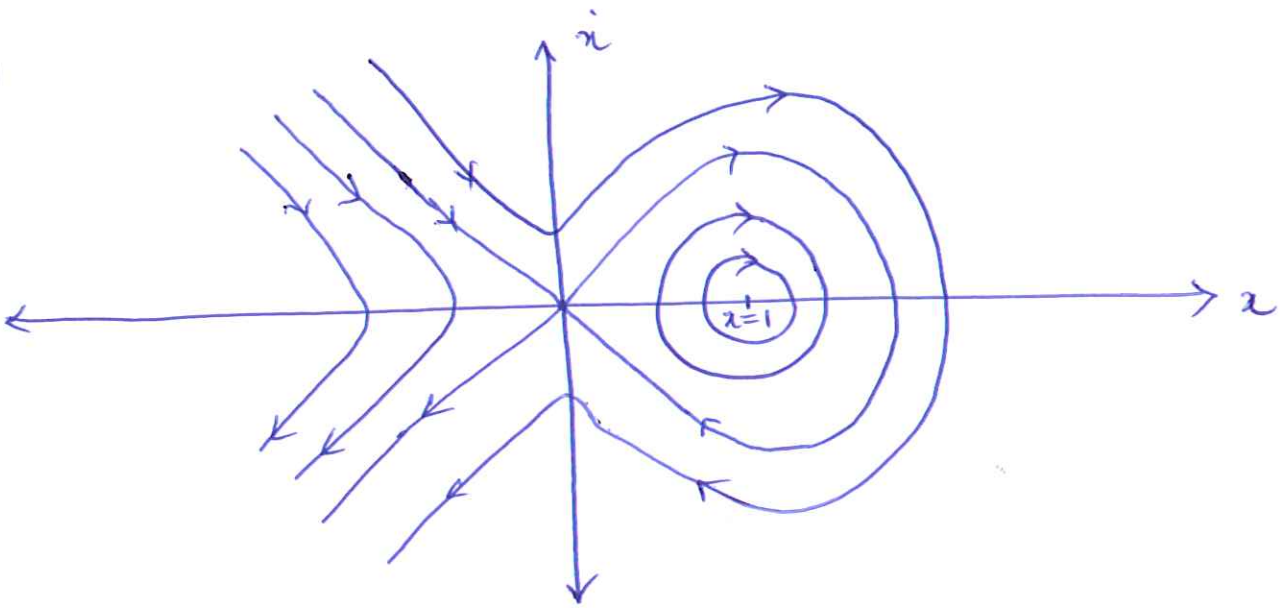
For singular point $(1,0)$ linearisation by introducing variable $z = x - 1 \Rightarrow \dot{z} = y$

and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and eigenvalues given by

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \text{ or } \lambda = \pm i$$

$\therefore$ centre.

(b)



(c) Equation of motion:

$$\ddot{x} - x + x^2 = 0$$

System is conservative $\frac{dV}{dx} = -x + x^2$

$$V(x) = -\frac{1}{2}x^2 + \frac{x^3}{3} + c \quad \text{(potential)}$$

(d) Conservation of energy implies

$$\frac{1}{2}\dot{x}^2 + V(x) = E \quad \text{(constant)}$$

$$\therefore \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{x^3}{3} = \text{constant}$$

satisfied by all points on trajectory including $(0,0)$ $\therefore$ constant $= 0$ and saddle point trajectory is

$$\frac{1}{2}\dot{x}^2 = \frac{1}{2}x^2 - \frac{x^3}{3}$$

$$\text{or} \quad y^2 = x^2 - \frac{2}{3}x^3 .$$

Q4

(a) Zeroth order solution given by:

$$x_0 = \underbrace{\frac{F_1}{p^2-\Omega_1^2}}_{A}\cos\Omega_1 t + \underbrace{\frac{F_2}{p^2-\Omega_2^2}}_{B}\cos\Omega_2 t$$

First order solution:

$$\ddot{x}_1 + p^2 x_1 = -\epsilon x_0^2 + F_1\cos\Omega_1 t + F_2\cos\Omega_2 t$$

$$\ddot{x}_1 + p^2 \dot{x}_1 = -\epsilon\left(A\cos\Omega_1 t + B\cos\Omega_2 t\right)^2 + F_1\cos\Omega_1 t + F_2\cos\Omega_2 t.$$

$$\ddot{x}_1 + p^2 x_1 = -\epsilon A^2\cos^2\Omega_1 t - \epsilon B^2\cos^2\Omega_2 t - 2\epsilon AB\cos\Omega_1 t\cos\Omega_2 t + F_1\cos\Omega_1 t + F_2\cos\Omega_2 t.$$

$$= -\epsilon A^2\left(\frac{1+\cos 2\Omega_1 t}{2}\right) - \epsilon B^2\left(\frac{1+\cos 2\Omega_2 t}{2}\right) - \frac{2\epsilon AB}{2}\left(\cos(\Omega_1+\Omega_2)t + \cos(\Omega_1-\Omega_2)t\right) + F_1\cos\Omega_1 t + F_2\cos\Omega_2 t$$

$$\therefore x_1 = \frac{F_1}{p^2-\Omega_1^2}\cos\Omega_1 t + \frac{F_2}{p^2-\Omega_2^2}\cos\Omega_2 t - \frac{\epsilon AB\cos(\Omega_1+\Omega_2)t}{p^2-(\Omega_1+\Omega_2)^2} - \frac{\epsilon AB\cos(\Omega_1-\Omega_2)t}{p^2-(\Omega_1-\Omega_2)^2} - \frac{\epsilon A^2\cos 2\Omega_1 t}{2(p^2-4\Omega_1^2)} - \frac{\epsilon B^2\cos 2\Omega_2 t}{2(p^2-4\Omega_2^2)} - \frac{\epsilon(A^2+B^2)}{2p^2}$$

(b) Harmonic Balance approach:

Assume solution as:

$$x = A\cos\Omega_1 t + B\cos\Omega_2 t + C\cos(2\Omega_1 t) + D\cos(2\Omega_2 t)$$
$$+ E\cos(\Omega_1+\Omega_2)t + F\cos(\Omega_1-\Omega_2)t + G$$

Plug back into solution to get:

$$-\Omega_1^2 A\cos\Omega_1 t - \Omega_2^2 B\cos\Omega_2 t - 4\Omega_1^2 C\cos 2\Omega_1 t$$
$$-4\Omega_2^2 D\cos\Omega_2 t - (\Omega_1+\Omega_2)^2 E\cos(\Omega_1+\Omega_2)t$$
$$-(\Omega_1-\Omega_2)^2 F\cos(\Omega_1-\Omega_2)t$$
$$+p^2 A\cos\Omega_1 t + p^2 B\cos\Omega_2 t + p^2 C\cos(2\Omega_1 t)$$
$$+p^2 D\cos(2\Omega_2 t) + p^2 E\cos(\Omega_1+\Omega_2)t + p^2 F\cos(\Omega_1-\Omega_2)t + p^2 G$$
$$+\epsilon\Big[A^2\left(\frac{1+\cos 2\Omega_1 t}{2}\right) + B^2\left(\frac{1+\cos 2\Omega_2 t}{2}\right) + C^2\left(\frac{1+\cos 4\Omega_1 t}{2}\right)$$
$$+D^2\left(\frac{1+\cos 4\Omega_2 t}{2}\right) + E^2\left(\frac{1+\cos 2(\Omega_1+\Omega_2)t}{2}\right) + F^2\left(\frac{1+\cos 2(\Omega_1-\Omega_2)t}{2}\right)$$
$$+G^2 + 2AB\left(\cos(\Omega_1-\Omega_2)t + \cos(\Omega_1+\Omega_2 t)\right)$$
$$+AC\left(\cos\Omega_1 t + \cos 3\Omega_1 t\right) + BC\left(\cos(\Omega_2+2\Omega_1)t + \cos(\Omega_2-2\Omega_1)t\right)$$
$$+AD\left(\cos(\Omega_1+2\Omega_1)t + \cos(\Omega_1-2\Omega_2)t\right) + BD\left(\cos\Omega_2 t + \cos 3\Omega_2 t\right)$$
$$+AE\left(\cos\Omega_2 t + \cos(2\Omega_1+\Omega_2)t\right) + BE\left(\cos\Omega_1 t + \cos(2\Omega_2+\Omega_1)t\right)$$
$$+AF\left(\cos(2\Omega_1-\Omega_2 t) + \cos\Omega_2 t\right) + BF\left(\cos\Omega_1 t + \cos(\Omega_1-2\Omega_2)t\right)$$
$$+2AG\cos\Omega_1 t + 2BG\cos\Omega_2 t + CD\left(\cos 2(\Omega_1+\Omega_2)t\right.$$
$$\Big]$$

(b) The resulting response consists of terms at the forcing frequencies Ω_1, Ω_2 as well as combination harmonics at frequencies $\Omega_1+\Omega_2, \Omega_1-\Omega_2, 2\Omega_1, 2\Omega_2$ and a DC term. This allows for a response to be generated at terms away from the forcing frequencies and the response amplitude can be elevated when the combination frequencies are in the neighbourhood of the natural frequency, p. This could be applied in cases requiring excitation of resonators at frequencies away from the natural frequency, to separate the response from the driving signals in the frequency domain eliminating any undesired direct coupling between the drive and the response.

Q1 Random excitation of non-linear systems (average 11/15)
This question was attempted by about half the students and generally well done as indicated by the high average mark for the question.

Q2 Circuit board impact during satellite launch (average 9.8/15)
This question was attempted by about half the students. Some students made errors deriving the transfer function in part (a); others who completed part (a) correctly had some difficulty calculating the probability of circuit board impact using the parameters provided though this was a relatively straightforward calculation.

Q3 Phase plane analysis for conservative systems (average 11.6/15)
The question was attempted by all students and generally well done. Most students were able to identify the singular points and establish their type. Most were able to provide a reasonable sketch of the phase portrait for the system as well. Some students had difficulties with part (d) in particular in establishing the equation for trajectories passing through the saddle point.

Q4 Method of iteration (average 11.3/15)
This question was attempted by all students and generally well done as indicated by the high average mark. Some students had difficulties with applying the method of iteration to arrive at a first-order solution for the response in (b); a number of students struggled to place their work in a practical context in (c).