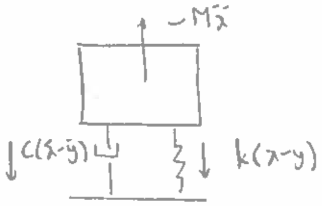
1 (a)



$$M\ddot{x} + C(\dot{x}-\dot{y}) + k(x-y) = 0$$

$$M\ddot{x} + C\dot{x} + kx = C\dot{y} + ky$$

Put $r = x-y \Rightarrow M\ddot{r} + C\dot{r} + kr = \underbrace{-M\ddot{y}}_{\text{white noise with spectrum } M^2S_{AA}}$

$\ddot{y}$: white noise with spectrum S_{AA}

From the data sheet $\sigma_r^2 = \dfrac{\pi(M^2S_{AA})}{ck}$

$$\sigma_{\dot{r}}^2 = \frac{\pi(M^2S_{AA})}{CM} = \frac{\pi M S_{AA}}{C}$$

Power dissipated $= C\dot{r}^2(t) \Rightarrow$ Average value $= C\sigma_{\dot{r}}^2 = \underline{\pi M S_{AA}}$. Independent of C and k and proportional to M [25%]

(b) $M\ddot{x} + C\dot{x} + kx = F(t)$

From data sheet $\sigma_x^2 = \dfrac{\pi S_{FF}}{Ck}$ $\sigma_{\dot{x}}^2 = \dfrac{\pi S_{FF}}{CM}$

Average power dissipated $= C\sigma_{\dot{x}}^2 = \underline{\dfrac{\pi S_{FF}}{M}}$ Independent of C and k and inversely proportional to M.

[15%]

(c) In the combined case $M\ddot{x} + C(\dot{x}-\dot{y}) + k(x-y) = F$

or $M\ddot{r} + C\dot{r} + kr = \underbrace{-M\ddot{r} + F}$

Statistically independent means that we can simply add the spo

$\Rightarrow S_0 = M^2S_{AA} + S_{FF}$

From the data sheet $\sigma_{\dot{r}}^2 = \dfrac{\pi S_0}{CM}$

Average power dissipated $= C\sigma_{\dot{r}}^2 = \pi\left\{MS_{AA} + \dfrac{S_{FF}}{M}\right\} = P$

For minimum $\dfrac{\partial P}{\partial M} = 0 \Rightarrow S_{AA} - \dfrac{S_{FF}}{M^2} = 0$

$\Rightarrow \underline{M = \left(\dfrac{S_{FF}}{S_{AA}}\right)^{1/2}}$ easy to check that the units are ok!

[20%]

(d) As noted in part (a), $\sigma_r^2 = \dfrac{\pi M^2 S_{AA}}{Ck}$

Now $r = x - y \Rightarrow x = r + y$

$$E[x^2] = E[r^2] + 2E[ry] + E[y^2]$$

$\downarrow$ This is infinite since $S_{yy} = \dfrac{1}{\omega^4} S_{AA}$

$$\sigma_y^2 = S_{AA}\int \frac{1}{\omega^4} d\omega = \infty$$

So the white noise approximation yields an infinite response σ_x^2. For the approximation to be valid, the actual spectrum must be reasonably flat over all frequencies where the response spectrum is significant. [20%]

(e) $M\ddot{x} + C\dot{x} + kx = F(t)$

$\Rightarrow \quad x(\omega) = H(\omega)F(\omega)$ where $H(\omega) = \dfrac{1}{-\omega^2 M + k + i\omega C}$

$\Rightarrow \quad \dot{x}(i\omega) = i\omega H(i\omega) F(\omega)$

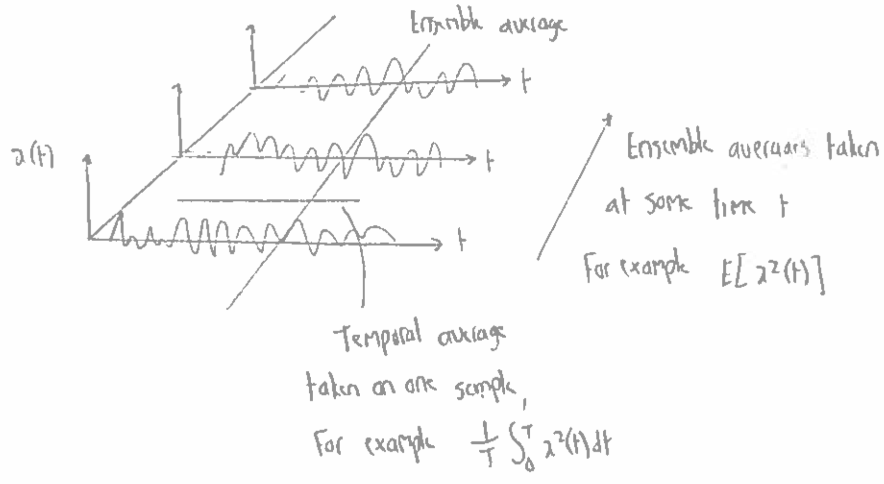
$$S_{\dot{x}\dot{x}}(\omega) = |i\omega H(\omega)|^2 S_{FF}(\omega)$$

$$= \frac{\omega^2 S_{FF}(\omega)}{(k - M\omega^2)^2 + (\omega C)^2}$$

Then power dissipated $= C\sigma_{\dot{x}}^2 = \underline{C\int_{-\infty}^{\infty} S_{\dot{x}\dot{x}}(\omega)d\omega}$

$\uparrow$ typically a numerical integral is needed [20%]

2 (a) Both terms are related to the notion of an <u>ensemble</u> of stochastic processes:



<u>Stationary</u> - ensemble averages are independent of time

<u>Ergodic</u> - ensemble averages are equal to temporal averages

Stationary but non-ergodic : eg, random sine wave $A\sin(\omega t+\varepsilon)$ [20%]

random variables across the ensemble

(b) For a Gaussian process $p(x,\dot{x}) = \left(\frac{1}{2\pi}\right)\left(\frac{1}{\sigma_x\sigma_{\dot{x}}}\right)e^{-\frac{1}{2}(x/\sigma_x)^2-\frac{1}{2}(\dot{x}/\sigma_{\dot{x}})^2}$

with $\sigma_x^2 = \int S_{xx}(\omega)\,d\omega$

$\sigma_{\dot{x}}^2 = \int \omega^2 S_{xx}(\omega)\,d\omega$

Process must be Gaussian and stationary - not met for a non-Gaussian process, for example [15%]
$x = y^3$, where y is Gaussian ; knowledge of $S_{xx}(\omega)$ does not give jpdf

(c) (i) In deriving the probability of failure, <u>level crossings are assumed to be independent</u>. This is most valid for a broad-band process, or for high levels.

(ii) In the derivation it is assumed that there is only <u>one peak per up-crossing</u>. This is most valid for a narrow band process. [20%]

(d) Let $\lambda(t)$ be the surface elevation

$$\sigma_\lambda^2 = \int S(\omega)d\omega = \int_0^\infty 225\omega\, e^{-8\omega^2} d\omega$$

$$= \left[\frac{-225}{2\times 8}\right]_0^\infty = 14.06 \Rightarrow \underline{\sigma_\lambda = 3.75}\ m$$

$$\sigma_{\dot\lambda}^2 = \int \omega^2 S(\omega)d\omega = \int_0^\infty 225\omega^3 e^{-8\omega^2} d\omega$$

$$= \int_0^\infty 225\omega^2 \frac{d}{d\omega}\left[\frac{-1}{16} e^{-8\omega^2}\right] d\omega$$

$$= \int_0^\infty \frac{225\times 2}{16}\, \omega\, e^{-8\omega^2}\, d\omega$$

$$= \frac{2}{16}\times 14.06 = 1.7575$$

$$\Rightarrow \underline{\sigma_{\dot\lambda} = 1.3257}\ m/s$$

From the data sheet $\nu_b^+ = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dot\lambda}}{\sigma_\lambda}\right) e^{-\frac{1}{2}(b/\sigma_\lambda)^2}$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{1.325}{3.75}\right) e^{-\frac{1}{2}(18/3.75)^2}$$

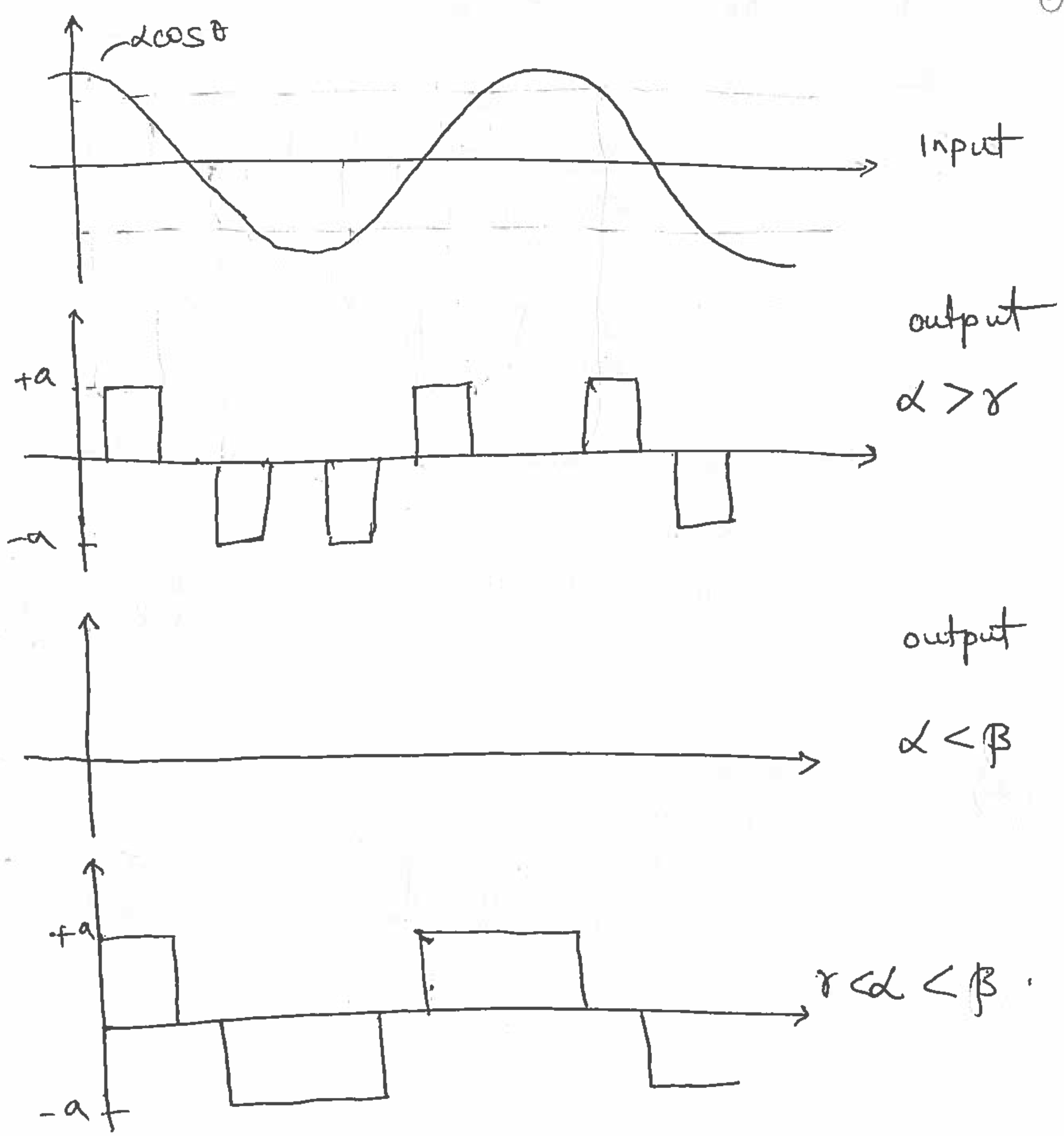
$$\underline{\nu_b^+ = 5.58\times 10^{-7}}\ sec^{-1}$$

Probability of crossing $= 1 - e^{-\nu_b^+ T} = 1 - e^{-5.58\times 10^{-7}\times 3\times 60\times 60}$

$$= \underline{0.006 = 0.6\%}$$

Too high given estimated 20 year life of the platform; the above figure is for a single 3 hour storm. [55%]

3 γ (a)



(b) D.F. for $\alpha > \gamma$

$$D = \frac{1}{\alpha\pi}\int_0^{2\pi} \text{output} \times \cos\theta \, d\theta$$

$$= \frac{4}{\alpha\pi}\int_{\theta_1}^{\theta_2} a\cos\theta \, d\theta$$

$$= \frac{4a}{\pi\alpha}\left[\sin\theta_2 - \sin\theta_1\right].$$

$$= \frac{4a}{\pi\alpha}\left[\sqrt{\left[1-\left(\frac{\beta}{\alpha}\right)^2\right]} - \sqrt{\left(1-\left(\frac{\gamma}{\alpha}\right)^2\right)}\right] \quad \text{—(1)}$$

D.F. for $\alpha < \beta = 0$ — (2)

For $\beta < \alpha < \gamma$

$$D.F. = \frac{4}{\alpha\pi}\int_0^{\theta_2}[a\cos\theta]\,d\theta$$

$$= \frac{4a}{\alpha\pi}\sqrt{1-\left(\frac{\beta}{\alpha}\right)^2} \quad \text{— (3)}$$

For the case when $\beta \to 0$ and $\gamma \to \infty$,

$0 < \alpha < \gamma$

$\therefore$ From (3) above D.F. $\to \frac{4a}{\pi\alpha}$ which is the case for switching control

(c) $m\ddot{x} + Dx \simeq f\cos\omega t$.

From (1) for $\alpha > \gamma$

$$-m\omega^2\alpha + \frac{4a}{\pi}\left[\sqrt{\left[1-\left(\frac{\beta}{\alpha}\right)^2\right]} - \sqrt{\left(1-\left(\frac{\gamma}{\alpha}\right)^2\right)}\right] \simeq f$$

4 (a)

$$\dot{u} = v$$
$$\dot{v} = +\cancel{\alpha} - u + \epsilon u^2$$

Singular or equilibrium points given as:

$$v = 0 \quad \text{and} \quad \cancel{\alpha} - u + \epsilon u^2 = 0$$

$$v = 0 \quad \text{and} \quad u = \frac{1 \pm \sqrt{1 \cancel{- 4\epsilon\alpha}}}{2\epsilon} = 0 \text{ or } \frac{1}{\epsilon}$$

(b) For $u = v = 0$;

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

eigenvalues given by

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \text{ or } \lambda = \pm i$$

$\therefore$ "centre"

For $u = \frac{1}{\epsilon}$ and $v = 0$;

$$z = \epsilon u - 1 \quad \text{and} \quad \dot{z} = \epsilon\dot{u} = \epsilon v$$

$$A = \begin{bmatrix} 0 & \epsilon \\ \frac{1}{\epsilon} & 0 \end{bmatrix}$$

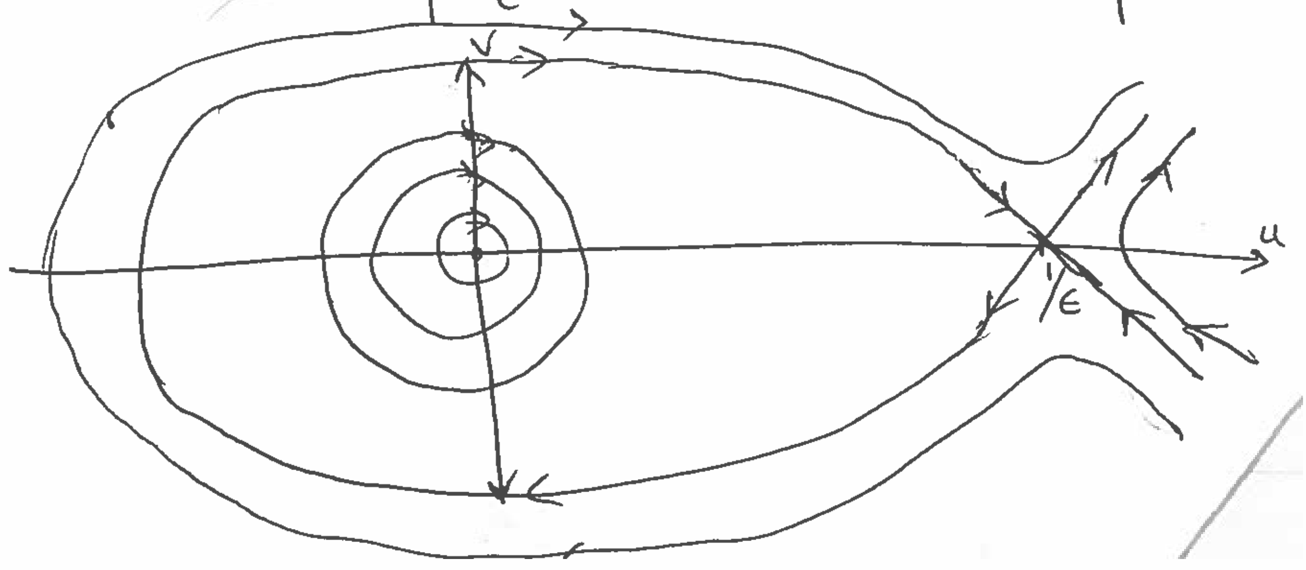
eigenvalues given by:

$$\begin{vmatrix} -\lambda & \epsilon \\ \frac{1}{\epsilon} & -\lambda \end{vmatrix} = 0 \text{ or } \lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$\therefore$ "saddle point".

(c)



(d) $u = A, \quad \dot{u} = 0 \quad$ at $t = 0$.

0th order solution

$$u = A\cos t$$

1st order:

$$\ddot{u} + u = \epsilon A^2 \cos^2 t$$

$$\ddot{u} + u = \frac{\epsilon A^2}{2}\left(1 + \cos 2t\right)$$

$$\therefore \quad u = C\cos t + \frac{\epsilon A^2}{2} + B\cos 2t$$

Equating $\cos 2t$ terms on LHS and RHS.

$$-3B = \frac{\epsilon A^2}{2} \Rightarrow B = -\frac{\epsilon A^2}{6}.$$

$$u = A \text{ at } t = 0 \Rightarrow C + B + \frac{\epsilon A^2}{2} = A.$$

$$\text{or} \quad C = A - \frac{\epsilon A^2}{2} + \frac{\epsilon A^2}{6} = A - \frac{\epsilon A^2}{3}.$$

$$\therefore \quad u = \left(A - \frac{\epsilon A^2}{3}\right)\cos t + \frac{\epsilon A^2}{2} \cancel{+} - \frac{\epsilon A^2}{6}\cos 2t$$

is the 1st order solution.

MODULE 4C7

Q1 Random vibration of an energy harvester
This question was attempted by most candidates and the students displayed a good understanding of the relevant principles and equations.

Q2 Descriptive questions plus crossing rates and failure probability
This question was relatively unpopular, being attempted by only 25% of the candidates. Students were possibly deterred by the initial descriptive part of the question. Most students who tackled the question produced good attempts at all parts.

Q3 Describing Functions
A popular question, with most students being able to derive the Describing Function for the three stated conditions. Lost marks were generally due to algebraic errors rather than failure to follow the correct procedure.

Q4 Singular points, phase portraits, and the method of iteration
This question was attempted by all of the candidates. Most marks were lost on the final part of the question, which involved the method of iteration. In most cases the problem was algebraic error, although in some cases the students failed to apply the method correctly.