Random and Non-linear Vibrations - Cribs 2019

4C7 CRIB

1\. (a) $x = Vt \Rightarrow v(t) = u(Vt) \Rightarrow E[v(t)v(t+\tau)] = E[u(Vt)u(Vt+V\tau)]$

$\uparrow$ wheel displacement

$= R_{uu}(V\tau)$

$$\underline{R_{vv}(\tau) = Ae^{-\alpha V|\tau|}}$$

$$S_{vv}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau} Ae^{-\alpha V|\tau|}\, d\tau$$

$$= \frac{A}{2\pi}\left\{\int_0^{\infty} e^{-i\omega\tau - \alpha V\tau}\, d\tau + \int_{-\infty}^{0} e^{-i\omega\tau + \alpha V\tau}\, d\tau\right\}$$

$$= \frac{A}{2\pi}\left\{\frac{1}{i\omega + \alpha V} - \frac{1}{i\omega - \alpha V}\right\} = \underline{\frac{A}{\pi}\left\{\frac{\alpha V}{\omega^2 + \alpha^2 V^2}\right\}} \quad [25\%]$$

(b) By inspection $M\ddot{y} + c(\dot{y} - \dot{v}) + k(y - v) = 0$

Put $r = y - v$, relative displacement

$$\Rightarrow M\ddot{r} + C\dot{r} + kr = M\ddot{v}$$

Assume $r = r(\omega)e^{i\omega t}$ and $v = v(\omega)e^{i\omega t}$ to give

$$r(\omega) = \left(\frac{-M\omega^2}{-M\omega^2 + Ci\omega + k}\right)v(\omega) = H(\omega)v(\omega)$$

$\uparrow$ frequency response function

$$\Rightarrow S_{rr}(\omega) = |H(\omega)|^2 S_{vv}(\omega)$$

Spring Force $F = kr \Rightarrow \underline{S_{FF}(\omega) = k^2|H(\omega)|^2 S_{vv}(\omega)}$ [30%]

(c) $M\ddot{r} + C\dot{r} + kr = M\ddot{v}$

$$\downarrow$$

$$\text{Spectrum} = M^2\omega^4 S_{vv}(\omega) = \frac{A}{\pi}\left\{\frac{M^2\omega^4\alpha V}{\omega^2+\alpha^2V^2}\right\}$$

Assume broad banded $\Rightarrow$ Replace with white noise at the resonant frequency

$$S_0 = \frac{A}{\pi}\left\{\frac{M^2\omega_n^4\alpha V}{\omega_n^2+\alpha^2V^2}\right\}$$

From data sheet $\sigma_r^2 = \dfrac{\pi S_0}{ck}$ $\quad \sigma_F^2 = k^2\sigma_r^2 = \pi k S_0/c$

Noting that $M^2\omega_n^4 = k^2$, then

$$\underline{\sigma_F^2 = \frac{A}{c}\left\{\frac{k^3\alpha V}{\omega_n^2+\alpha^2V^2}\right\}}$$

[25%]

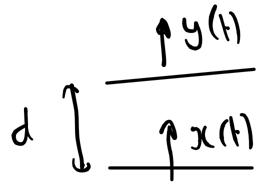
(d) From part (b), if $M\to\infty$ then $H(\omega)\to 1$ and $S_{FF}(\omega)\to k^2S_{vv}(\omega)$

This makes sense, because the mass will not move as $M\to\infty$ and we will have $r(t)=v(t)$. In this case $\sigma_F^2 = k^2\sigma_v^2 = \underline{k^2A}$

White noise approximation breaks down because $H(\omega)=1$ is very <u>broad</u> and the input spectrum cannot be broader than this!

[20%]

2.(a)



Put $r = x(t) - y(t)$

Impact will occur if $r \geq d \Rightarrow$ level crossing problem

$$r^2 = x^2 - 2xy + y^2 \Rightarrow \sigma_r^2 = \sigma_x^2 + \sigma_y^2 = 0.6^2 + 0.8^2 = 1$$

$$\dot{r}^2 = \dot{x}^2 - 2\dot{x}\dot{y} + \dot{y}^2 \Rightarrow \sigma_{\dot{r}}^2 = \sigma_{\dot{x}}^2 + \sigma_{\dot{y}}^2 = 122^2 + 276^2 = 89960$$

$$\Rightarrow \quad \sigma_r = 1 \quad \text{and} \quad \sigma_{\dot{r}} = 299.9$$

$$\nu_d^+ = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{r}}}{\sigma_r}\right) e^{-\frac{1}{2}(d/\sigma_r)^2} = \left(\frac{1}{2\pi}\right)\left(\frac{299}{1}\right) e^{-\frac{1}{2}(5/1)^2} = 1.779\times10^{-4}$$

$$\text{Probability of impact} = 1 - e^{-\nu_d^+ T} = 1 - e^{-1.779\times10^{-4}\times 3\times 60} = \underline{0.0315}$$

[30%]

(b) Excitation broad banded $\Rightarrow$ response similar to the response of an oscillator to white noise

From the data sheet $\sigma_x^2 = \dfrac{\pi S_0}{ck}$, $\sigma_{\dot{x}}^2 = \dfrac{\pi S_0}{cM}$ $\Rightarrow$ m.s. response $\propto \left(\frac{1}{c}\right)$

Double damping $\Rightarrow$ $\sigma_r \rightarrow 1/\sqrt{2}$

$\sigma_{\dot{r}} \rightarrow 299.9/\sqrt{2}$

$$\Rightarrow \quad \nu_d^+ = \left(\frac{1}{2\pi}\right)\left(\frac{299}{1}\right) e^{-\frac{1}{2}(\sqrt{2}\times5)^2} = 6.63\times10^{-10}$$

$$P = 1 - e^{-\nu_d^+ T} = 1 - e^{-6.63\times10^{-10}\times3\times60} = \underline{1.19\times10^{-7}}$$

[35%]

This is a reasonable probability of failure

(c) From the data sheet $D = \nu_0^+ T \, E[1/N(s)]$; $N(s) = 4000 \, S^{-4}$

$$1/N(s) = \frac{0.3}{4000} \times b$$

$$\downarrow$$

$$\int_0^\infty \frac{1}{N(s)} p(b)\,db$$

$$\downarrow$$

peak distribution $= \frac{b}{\sigma_r^2} e^{-\frac{1}{2}(b/\sigma_r)^2}$

From data sheet

So $D = \nu_0^+ T \left(\frac{0.3}{4000\,\sigma_r^2}\right) \underbrace{\int_0^\infty b^2 e^{-\frac{1}{2}(b/\sigma_r)^2} db}_{\frac{1}{2} \times \sqrt{2\pi} \times \sigma_r^3}$

Many students used a Gaussian distribution here by mistake. The peaks are Rayleigh

So $D = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dot{r}}}{\sigma_r}\right) T \left(\frac{0.3}{4000}\right) \sqrt{\frac{\pi}{2}}\, \sigma_r$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{1.22}{0.6}\right) 3 \times 60 \left(\frac{0.3}{4000}\right) \sqrt{\frac{\pi}{2}} \times 0.6 = \underline{0.3285}$$

[35%]

This is $\underline{\text{too high}}$ because a Factor of safety of between 4 and 5 must be applied.

3.(a) $\ddot{x} - \varepsilon\dot{x}(3-8x^4) + x = 0$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ 3\varepsilon\dot{x} - x + 8\varepsilon\dot{x}x^4 \end{pmatrix} \Rightarrow \text{equilibrium point at } (0,0)$$

Consider stability by looking at motion around $(0,0)$

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 3\varepsilon \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

Assume dependency $e^{\lambda t}$ $\Rightarrow$ $\begin{vmatrix} -\lambda & 1 \\ -1 & 3\varepsilon-\lambda \end{vmatrix} = 0$

$$\Rightarrow \quad -\lambda(3\varepsilon-\lambda)+1=0 \Rightarrow \lambda^2 - 3\varepsilon\lambda + 1 = 0$$

$$\Rightarrow \quad \lambda = \tfrac{1}{2}\Big\{ \underbrace{3\varepsilon}_{\text{positive}} \pm \underbrace{\sqrt{9\varepsilon^2-4}}_{\text{complex}} \Big\} \Rightarrow \text{conjugate pair with positive real part}$$

$\Rightarrow$ <u>unstable focus</u>

(b) $\ddot{x} + x = \varepsilon\dot{x}(3-8x^4)$

Zero order solution $\quad \ddot{x}_0 + x_0 = 0 \Rightarrow x_0 = A\cos t$

1st order solution $\quad \ddot{x}_1 + x_1 = -A\varepsilon\sin t\,(3-8A^4\cos^4 t)$

$$\cos^4 t \downarrow \tfrac{1}{4}(1+\cos 2t)^2$$

$$\downarrow \tfrac{1}{4}(1+2\cos 2t+\cos^2 2t)$$

$$\downarrow \tfrac{1}{4}(1+2\cos 2t+\tfrac{1}{2}+\tfrac{1}{2}\cos 4t)$$

$$\tfrac{3}{8} + \tfrac{1}{2}\cos 2t + \tfrac{1}{8}\cos 4t \leftarrow \tfrac{1}{4}(1+2\cos 2t+\tfrac{1}{2}+\tfrac{1}{2}\cos 4t)$$

$$\ddot{x}_1 + x_1 = -3A\varepsilon \sin t + 8\varepsilon A^5 \left[\tfrac{3}{8}\sin t + \tfrac{1}{2}\cos 2t \sin t + \tfrac{1}{8}\cos 4t \sin t \right]$$

$$= (-3A\varepsilon + 3\varepsilon A^5)\sin t + 8\varepsilon A^5 \left[\tfrac{1}{4}\sin 3t - \tfrac{1}{4}\sin t + \tfrac{1}{16}\sin 5t - \tfrac{1}{16}\sin 3t \right]$$

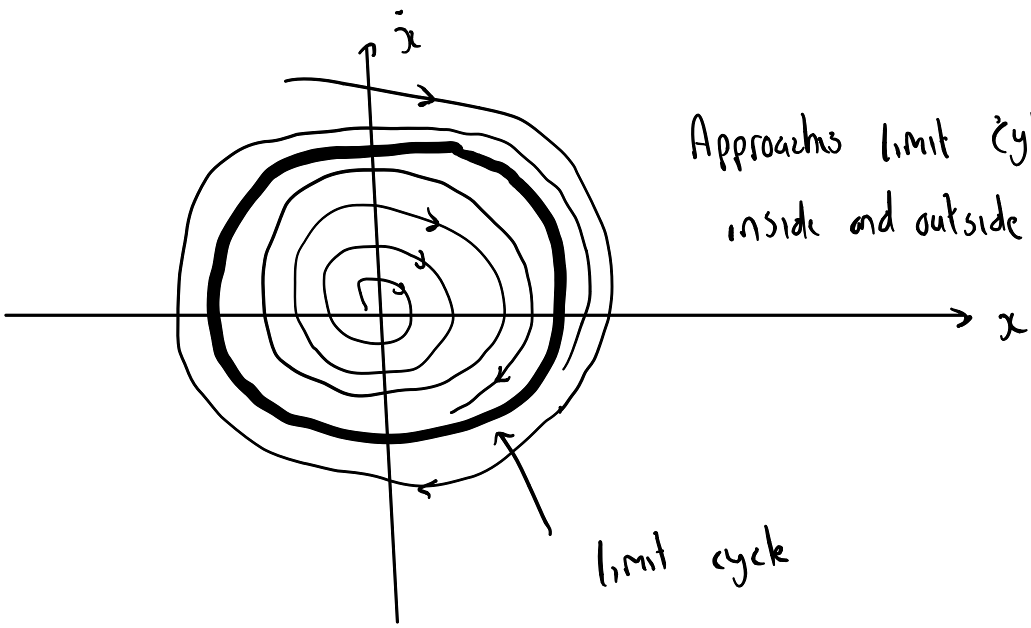
$$= \underbrace{(-3A\varepsilon + \varepsilon A^5)}\sin t + 8\varepsilon A^5 \left[\tfrac{3}{16}\sin 3t + \tfrac{1}{16}\sin 5t \right]$$

To avoid resonance need $\underline{A^4 = 3}$

Then $\ddot{x}_1 + x_1 = \frac{1}{2}\varepsilon A^5 \left[3\sin 3t + \sin 5t \right]$

$$\Rightarrow \underline{x_1 = \tfrac{1}{2}\varepsilon A^5 \left[-\tfrac{3}{8}\sin 3t - \tfrac{1}{24}\sin 5t \right]}$$ [50%]

(c)



Approaches limit cycle from inside and outside

[30%]

4. a) $\ddot{x} + p^2 x + \mu x^3 = k\cos\omega t$

For describing function $\int_0^T \mu x^3 \times x \, dt = \int_0^T D x \times x \, dt$ with $x = A\cos\omega t$

$$\Rightarrow \mu A^4 \int_0^T \cos^4\omega t \, dt = DA^2 \int_0^T \cos^2\omega t \, dt$$

$$\mu A^4 \int_0^T \left[\tfrac{1}{2}(1+\cos 2\omega t)\right]^2 dt = \tfrac{1}{2} DA^2 T$$

$$\mu A^4 \int_0^T \left[\tfrac{1}{4} + \tfrac{1}{2}\cos 2\omega t + \tfrac{1}{4}\cos^2 2\omega t\right] dt = \tfrac{1}{2} DA^2 T$$

$$\Rightarrow T\mu A^4 \left[\tfrac{1}{4} + \tfrac{1}{8}\right] = \tfrac{1}{2} DA^2 T \quad \Rightarrow \quad \underline{D = \tfrac{3}{4}\mu A^2}$$

$$\Rightarrow \quad \underline{(-\omega^2 + p^2 + \tfrac{3}{4}\mu A^2) A = k}$$

[30%]

b) $\ddot{x} + p^2 x + \mu x^3 = k\cos\omega t$

$$\cancel{\ddot{x}_0} + \ddot{u} + \cancel{p^2 x_0} + p^2 u + \mu(\cancel{x_0^3} + 3x_0^2 u + \underbrace{3x_0 u^2 + u^3}_{\text{small}}) = \cancel{k\cos\omega t}$$

$$\Rightarrow \quad \underline{\ddot{u} + (p^2 + 3\mu x_0^2) u = 0}$$

[20%]

c) $x_0 = A\cos\omega t \Rightarrow \ddot{u} + (p^2 + 3\mu A^2 \cos^2\omega t) u = 0$

$$\Rightarrow \ddot{u} + (p^2 + \tfrac{3}{2}\mu A^2 + \tfrac{3}{2}\mu A^2 \cos 2\omega t) u = 0$$

$$\Rightarrow \underline{\ddot{u} + (P^2 + \varepsilon\cos\Omega t) u = 0} \leftarrow \text{Mathieu equation}$$

$P^2 = p^2 + \tfrac{3}{2}\mu A^2 \quad \varepsilon = \tfrac{3}{2}\mu A^2 \quad \Omega = 2\omega$

First order approximation $\ddot{u} + p^2 u = 0 \Rightarrow u_0 = A\cos pt$

Second order approximation $\ddot{u}_1 + p^2 u_1 = -\varepsilon A \cos\Omega t \cos pt$

$$= -\frac{\varepsilon A}{2}\left[\cos(\Omega+p)t + \cos(\Omega-p)t\right]$$

Resonance when $p = \Omega + p \rightarrow$ not a physical solution

or $p = \Omega - p \Rightarrow \underline{\Omega = 2p}$

In the present case $2\omega = 2\sqrt{p^2 + \tfrac{3}{2}NA^2}$

$$\Rightarrow \underline{\omega = \sqrt{p^2 + \tfrac{3}{2}NA^2}}$$

[50%]

4C7: Examiner's comments:

Q1 Spectral analysis of vehicle dynamics
No major conceptual difficulties were evident, and the question was generally answered well.

Q2 Circuit board impact and fatigue
Parts (a) and (b), concerning crossing-rates and the probability of impact, were well done. In Part (c), concerning fatigue damage, only one of the 17 students who attempted the question obtained the correct numerical answer. A common error was to use a Gaussian distribution for the stress peaks, instead of a Rayleigh distribution.

Q3 Limit cycle oscillations
Part (a) was very well done, but the detailed algebra (trigonometry) in section (b) defeated many students. Most students made a reasonable attempt at the phase plane portrait in Part (c), even if Part (b) had not been completed.

Q4 Mathieu equation
This was the least popular question (8 attempts), although the average mark was reasonably good (60%). Part (b) was not well done, which was surprising given that the target formula was given on the exam paper and the question involved a straight forward substitution followed by the neglect of small terms.