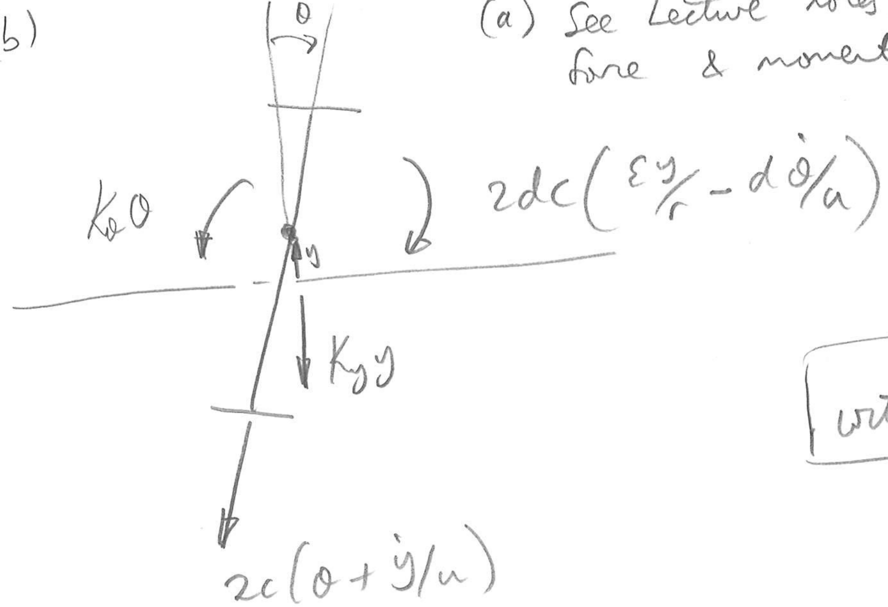
ENGINEERING TRIPOS PART IIB PAPER 4C8 SOLUTIONS, 2015

1 (b)

(a) See Lecture notes for derivation of rel. force & moment.



$$\boxed{\text{with } K_\theta = K_y a^2}$$

(50%)

$$\sum \underline{F} = 0: \quad 2c\left(\theta + \frac{\dot{y}}{u}\right) + K_y y = 0 \qquad \text{①}$$

$$\circlearrowright \sum \underset{\sim}{M} = 0: \quad 2dc\left(\varepsilon y/r - d\dot{\theta}/u\right) - K_\theta \theta = 0 \qquad \text{②}$$

$$\text{①} \rightarrow \theta = -\frac{K_y}{2c} y - \dot{y}/u$$
$$\& \quad \dot{\theta} = -\frac{K_y}{2c}\dot{y} - \ddot{y}/u \qquad \text{③}$$

③ into ② gives

$$\frac{\varepsilon y}{r} + \frac{d}{u}\left(+\frac{K_y}{2c}\dot{y} + \ddot{y}/u\right) + \frac{a^2 K_y}{2dc}\left(+\frac{K_y}{2c} y + \dot{y}/u\right) = 0$$

$$\ddot{y}\left(\frac{d}{u^2}\right) + \dot{y}\frac{K_y}{2dcu}\left(d^2 + a^2\right) + y\left(\frac{\varepsilon}{r} + \frac{a^2 K_y^2}{4dc^2}\right) = 0$$

$$\text{ie} \quad \ddot{y} + \dot{y}\left[\frac{uK_y(d^2+a^2)}{2d^2c}\right] + y\left[\frac{u^2}{d}\left(\frac{\varepsilon}{r} + \frac{a^2K_y^2}{4dc^2}\right)\right] = 0.$$

(30%)

(c) Hunting wavelength

$$\omega_n^2 = \frac{u^2}{d}\left(\varepsilon/r + \frac{a^2 K_y^2}{4dc^2}\right)$$

$$\lambda = \frac{2\pi u}{\omega_n} = \frac{2\pi \cancel{u}}{\cancel{u}\sqrt{\dfrac{\varepsilon}{dr} + \dfrac{a^2 K_y^2}{4d^2c^2}}}$$

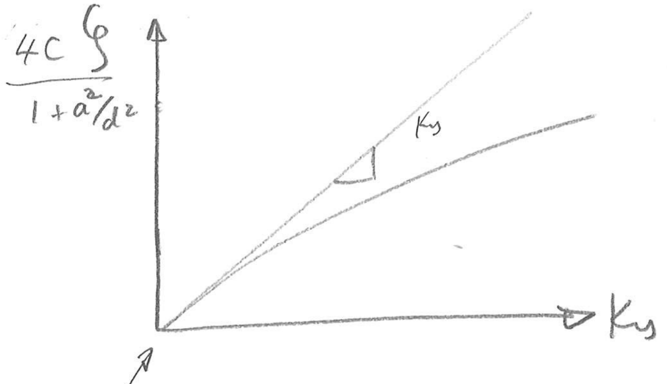
For zero suspension stiffness, $K_y = 0 \Rightarrow \lambda = 2\pi\sqrt{\dfrac{dr}{\varepsilon}}$ as per free wheelset

(10%) ✓

(d) Damping ratio of hunting mode

$$\zeta = \frac{\cancel{u} K_y (d^2 + a^2)}{4d^2 C\sqrt{\dfrac{\cancel{u^2}}{d}\left(\dfrac{\varepsilon}{r} + \dfrac{a^2 K_y^2}{4dc^2}\right)}}$$

$$= \frac{1 + a^2/d^2}{4C}\left[\frac{K_y}{\sqrt{\dfrac{\varepsilon}{dr} + \dfrac{a^2 K_y^2}{4d^2c^2}}}\right]$$



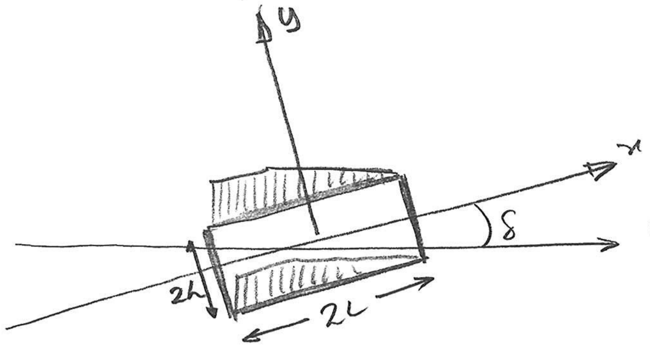
zero for $K_y = 0$ as per free wheelset

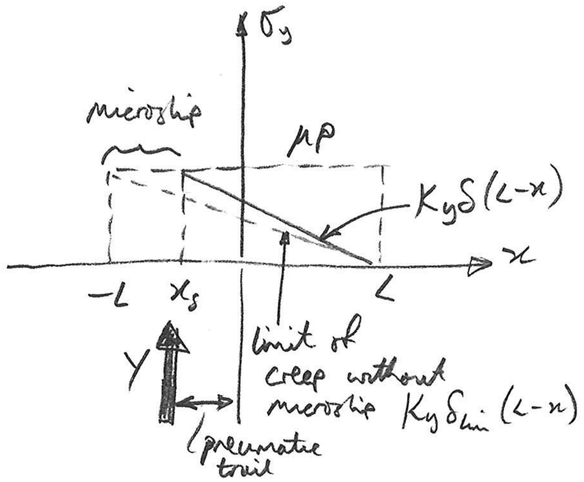
(independent of speed)

(10%)

PART II 4C8 EXAM 2015 — SOLUTIONS

2.





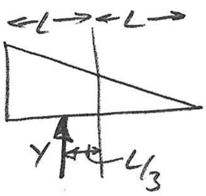
(a) Small δ — no microslip:

Displacement of bristles is $q_y = \delta(L-x)$ — (1)

No longitudinal slip ∴ $q_x = 0 \Rightarrow \sigma_x = 0$

Brush model (data sheet) $\sigma_y = k_y q_y = K_y \delta (L-x)$ — (2)

Realigning moment (data sheet):



$$N = \iint_A (x\sigma_y - y\sigma_x)\,dA = \int_{-h}^{h}\int_{-L}^{L} x K_y \delta (L-x)\,dx = -\frac{4}{3}L^3 h K_y \delta$$ — (3)

Pneumatic trail $l = N/Y$ = distance of centre of pressure from centre of contact patch $\Rightarrow l = L/3$ [30%]

(b) The region of microslip starts at the location in the contact patch where

$$\sigma_y = \mu p = K_y \delta (L-x) \Rightarrow x_s = L - \frac{\mu p}{K_y \delta}$$ — (4)

Once microslip begins, (3) has to be evaluated in 2 parts

$$N = 2h\int_{-L}^{x_s} x \mu p\,dx + 2h\int_{x_s}^{L} x K_y \delta (L-x)\,dx$$ — (5)

Integrate (5) and use $p = \frac{Z}{4hL}$ & $\lambda = \frac{4L^2 h K_y}{\mu Z}\delta$ — (6)

To be consistent with part (a), the two solutions should have the same values of $N(\delta_{lim})$ and $\left.\frac{dN}{d\delta}\right|_{\delta_{lim}}$ where δ_{lim} is the steer angle at which microslip first starts at the rear of the contact area.

1 cont

δ_{lim} is found from (2) as:

$$\left| K_y \delta_{lim} (L - -L) \right| = \mu P \quad \text{ie} \quad \delta_{lim} = \frac{\mu (Z/4Lh)}{2 K_y L}$$

$$\Rightarrow \delta_{lim} = \frac{\mu Z}{8 L^2 h K_y} \Rightarrow \lambda_{lim} = 1/2 \qquad (7)$$

from (3) $N(\delta_{lim}) = -\frac{4}{3} L^3 h K_y \left(\frac{\mu Z}{8 L^2 h K_y} \right) = -\frac{\mu Z L}{6}$

From (6) $\lambda_{lim} = \frac{4 L^2 h K_y}{\mu Z} \left(\frac{\mu Z}{8 L^2 h K_y} \right) = 1/2$

& from Question sheet $\frac{N}{\mu Z L} = \frac{1}{4\lambda}\left(\frac{1}{3\lambda} - 1 \right) = \frac{1 - 3\lambda}{12 \lambda^2}$ (8

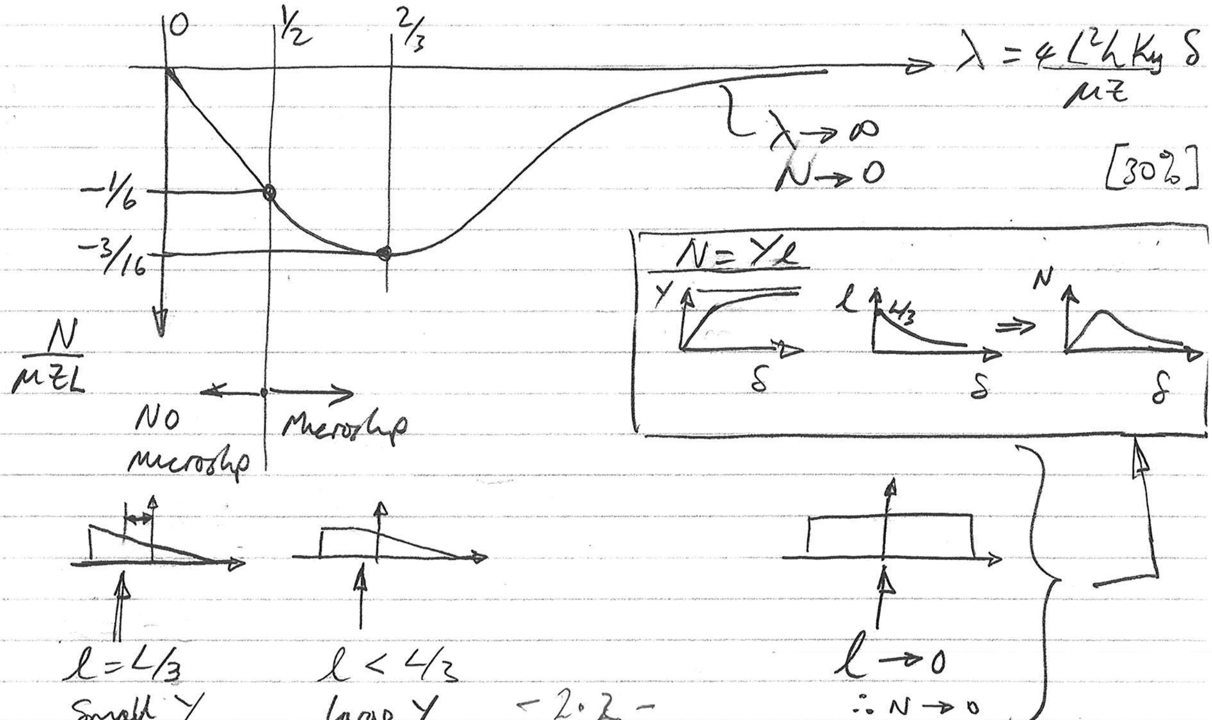
$$\frac{N(\delta_{lim})}{\mu Z L} = \frac{1}{2}\left(2/3 - 1 \right) = -1/6 \checkmark$$

[20%]

(c) N_{max} found from

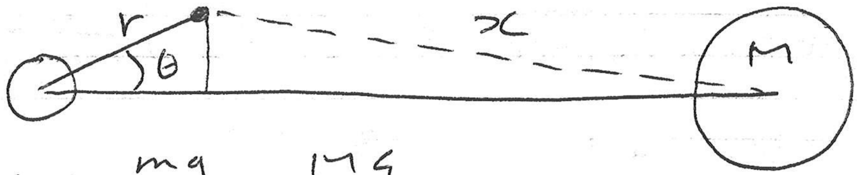
$$\frac{1}{\mu Z L}\left(\frac{dN}{d\lambda} \right) = \frac{12 \lambda^2 (-3) - (1 - 3\lambda) 24 \lambda}{144 \lambda^4} = 0 \Rightarrow \lambda = 2/3$$

$$\Rightarrow \frac{N_{max}}{\mu Z L} = -3/16$$



1 cont (d) Realigning moment is usually neglected in simple models of vehicle dynamics because the pneumatic trail is small relative to length of the vehicle. ie N is small relative to yawing moments on the vehicle body due to Y. [20%]

3 a)



$$U = \frac{mg}{r} + \frac{MG}{x}$$

but $x = \sqrt{(R - r\cos\theta)^2 + (r\sin\theta)^2}$

$$= \sqrt{R^2 - 2Rr\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= R\left(1 + \frac{r^2}{R^2} - 2\frac{r}{R}\cos\theta\right)^{\frac{1}{2}}$$

$$\approx R\left(1 - \frac{r}{R}\right)\cos\theta \quad \text{since } r << R$$

$$\approx \frac{R}{(1 + r/R\cos\theta)}$$

So $\underline{U = \frac{mG}{r} + \frac{MG}{R}\left(1 + \frac{r}{R}\cos\theta\right)}$

$$F_R = \frac{\partial U}{\partial r} = -\frac{\mu}{r^2} + \frac{MG}{R}\left(1 + \frac{\cos\theta}{R}\right)$$

So $\underline{\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} + \frac{MG}{R}\left(1 + \frac{\cos\theta}{R}\right)}$

$$F_\theta = \frac{1}{r}\frac{\partial U}{\partial \theta} = -\frac{MG}{R^2}\sin\theta$$

So $\ddot{r} - r\dot{\theta}^2 = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -\frac{MG}{r^2}\sin\theta$

whence $\underline{\frac{d}{dt}(r^2\dot{\theta}) = -\frac{MGr}{R^2}\sin\theta}$

b) If $\theta = \omega t + \alpha$ and $\alpha << 1$, then

$\sin\theta$

will be dominated by ωt.

Hence $\frac{d}{dt}(r^2\dot{\theta}) = -\frac{MGr}{R^2}\sin\omega t$

So $r^2\dot{\theta} = \frac{MGr}{R^2}\cos\omega t + C$

The constant part on the LHS is $a^2\omega$

So $C = a^2\omega = h_0$

Also, $\frac{MGr}{R^2} \simeq \frac{MGa}{R^2}$

So $\underline{r^2\dot{\theta} \simeq h_0 + \frac{MGa}{R^2}}$

c) To solve $\ddot{\delta} + \omega^2\delta = \frac{3MG}{R^2}\cos\omega t$

For C.F., let $\delta = A\cos\omega t + B\sin\omega t$

For P.I., try $\delta = Ct\sin\omega t$

$$\dot{\delta} = C\sin\omega t + C\omega t\cos\omega t$$

$$\ddot{\delta} = 2C\omega\cos\omega t - C\omega^2 t\sin\omega t$$

$\therefore\ \ddot{\delta} + \omega^2\delta = 2C\omega\cos\omega t$

So $2C\omega = \frac{3MG}{R^2} \Rightarrow C = \frac{3MG}{2\omega R^2}$

$\therefore\ \underline{\delta = A\cos\omega t + B\sin\omega t + \frac{3MG}{2\omega R^2}t\sin\omega t}$

δ will grow with time, which may make the orbit unstable - but the sun doesn't stay at $(R,0)$ long enough for this to happen

4a) At perigee of probe's orbit, radius & velocity are r_0 & V. If values at apogee are r_2 & V_2,

$$V_2 r_2 = V r_0$$

$$\tfrac{1}{2} V_2^2 - \frac{\mu}{r_2} = \tfrac{1}{2} V^2 - \frac{\mu}{r_0}$$

$$\rightarrow V_2^2 = V^2 - 2\mu\left(\frac{r_2 - r_0}{r_2 r_0}\right)$$

So $V^2 r_2^2 - 2\mu r_2^2\left(\frac{r_2 - r_0}{r_2 r_0}\right) = V^2 r_0^2$

$$V^2(r_2^2 - r_0^2) r_2 r_0 = 2\mu r_2^2 (r_2 - r_0)$$

$$V^2(r_2 + r_0) r_2 r_0 = 2\mu r_2^2$$

$$\frac{r_2^2 r_0 + r_2 r_0^2}{r_2^2} = \frac{2\mu}{V^2}$$

$$\frac{r_0^2}{r_2} = \frac{2\mu}{V^2} - r_0$$

$$\therefore \underline{r_2 = \frac{r_0^2}{2\mu/V^2 - r_0}} \qquad (1)$$

From data sheet, $r = \frac{L}{1 + e\cos\theta}$

So $\frac{r_2}{r_0} = \frac{1+e}{1-e}$

$$r_2 - r_2 e = r_0 + r_0 e$$

$$(r_2 + r_0) e = r_2 - r_0$$

$$e = \frac{r_2 - r_0}{r_2 + r_0} = \frac{r_0^2 - r_0\left(\frac{2\mu}{V^2} - r_0\right)}{r_0^2 + r_0\left(2\mu/V^2 - r_0\right)} \quad \text{from (1)}$$

$$\therefore e = \frac{2r_0 - \frac{2\mu}{V^2}}{2\mu/V^2} = \underline{\underline{\frac{V^2 r_0 - \mu}{\mu}}} \qquad (2)$$

and $a = \dfrac{r_0 + r_2}{2} = \dfrac{r_0^2 + r_0\left(\frac{2\mu}{V^2} - r_0\right)}{2\left(\frac{2\mu}{V^2} - r_0\right)}$

$$= \frac{r_0^2 V^2 + r_0(2\mu - r_0 V^2)}{4\mu - 2r_0 V^2}$$

$$= \frac{\mu r_0}{2\mu - r_0 V^2}$$

b) Initial velocity is V_0, such that

$\dfrac{V_0^2}{r_0} = \dfrac{\mu}{r_0^2}$ i.e. $V_0 = \sqrt{\mu / r_0}$

V is such that $e = 1$ so from (2)

$V^2 r_0 = 2\mu$, i.e. $V = \sqrt{2}\sqrt{\mu/r_0}$

If launcher gets new velocity V_1, say, use (1) to say

$$r_E = \frac{r_0^2}{\frac{2\mu}{V_1^2} - r_0}$$

$$\therefore \frac{2\mu}{V_1^2} = \frac{r_0^2}{r_E} + r_0 = r_0\left(\frac{r_0}{r_E} + 1\right)$$

$$\therefore V_1 = \sqrt{\frac{2\mu}{r_0(r_0/r_E + 1)}}$$

By momentum, $4V_0 = V + 3V_1$

So $4\sqrt{\mu/r_0} = \sqrt{2}\sqrt{\mu/r_0} + 3\sqrt{\dfrac{2\mu}{r_0(r_0/r_E + 1)}}$

$$\left(4-\sqrt{2}\right)\sqrt{\mu/r_0} = 3\sqrt{\mu/r_0}\,\sqrt{\frac{2}{r_0/r_E+1}}$$

$$\frac{r_0/r_E+1}{2} = \left(\frac{3}{4-\sqrt{2}}\right)^2 = 1.3460$$

$$r_0/R_E = 1.6921$$

So height of orbit $= 0.6921\, r_E = \underline{\underline{4414 \text{ km}}}$

Prof D Cebon

ENGINEERING TRIPOS PART IIB 2015
MODULE 4C8: APPLICATIONS OF DYNAMICS

Comments on exam questions

Question 1
Railway wheelset hunting. Attempted by most candidates. The derivation in part (a) was standard and generally well done. In part (b), quite a few candidates put the two simple (coupled) equations of motion into matrix form and launched into Routh Hurwitz calculations, when all that was needed was to combine them into a single DoF oscillator, as per similar examples in the lecture notes.

Question 2
Tyre realigning moment: Not well done. Candidates could plug into the datasheet formula to calculate the simple result in part (a), but they generally didn't know the term 'pneumatic trail' which was explained in lectures. Part (b) was a mess. Candidates tried to derive the formula, instead of explaining the derivation and very few showed that the linear and nonlinear formulae agreed at their intersection point. Answers to part (d) were all over the place… lots of guesses: again indicating that the lecture notes had not been read.

Question 3
Candidates who attempted this question generally knew what to do, though there were some rather unconvincing explanations of the approximations needed to obtain the answer for part (a). Part (b) was well done, but the only a few candidates identified the need for a complimentary function as well as a particular integral for part (c). Of those who did, the majority then realised that the particular integral implies a perturbation which grows with time, and commented appropriately.

Question 4
Q4: This was the more accessible of the two questions on orbital mechanics, so it possibly attracted those candidates who had not given this subject their full attention. Some candidates answered part (a) very efficiently by using the orbital data given in the Part I Mechanics data book; at the other end of the spectrum, some were clearly unsure what basic equations to use, to solve the stated problem. Several candidates made a good attempt at part (b), but only one candidate was able to solve it correctly.