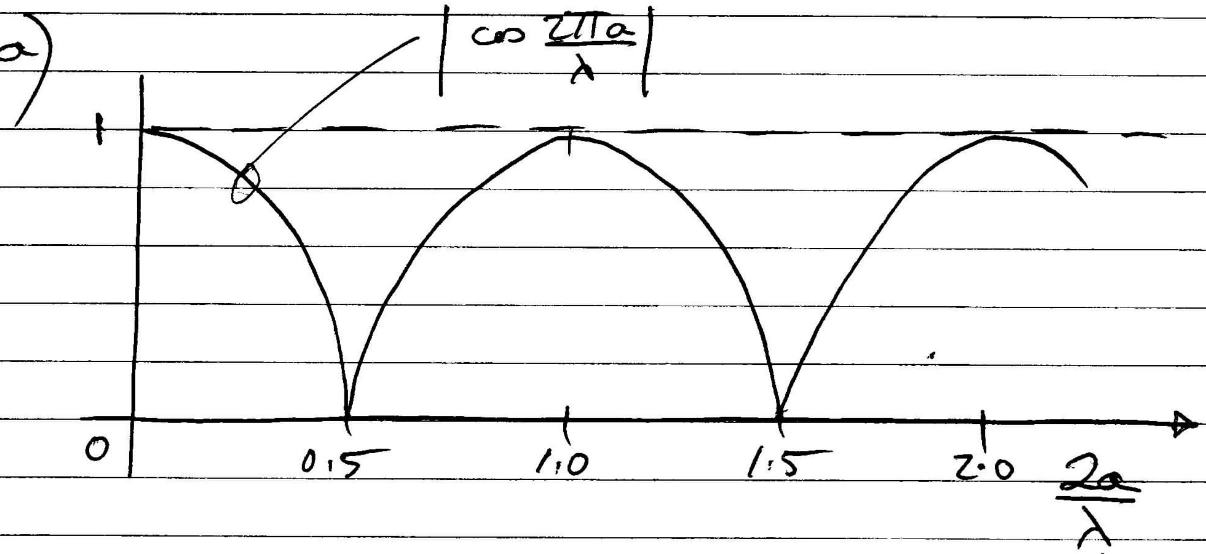
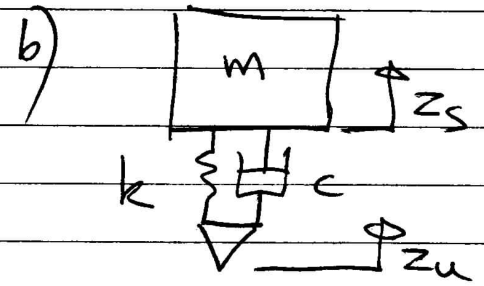
Dr D J Cole

Engineering Tripos Part IIB 2016
Module 4C8 Applications of Dynamics
Solutions

1 a)



A two-axle vehicle ~~travelling~~ along a sinusoidal profile of wavelength λ will experience pure bounce excitation if a whole number of wavelengths $n\lambda$ fit with the wheelbase L. If there is a whole number plus a half wavelength $(n+\frac{1}{2})\lambda$ in the wheelbase L then the vehicle experiences pure pitch excitation and no bounce excitation. The corresponding excitation frequencies f depend on vehicle speed U according to $U=f\lambda$. Thus the frequencies at which there is strong bounce excitation are nU/L, and strong pitch excitation at $(n+\frac{1}{2})U/L$. For a vehicle with suspension the wheelbase filtering leads to the vibration response of the spring mass being strongly dependent on the speed of the vehicle.

b) 

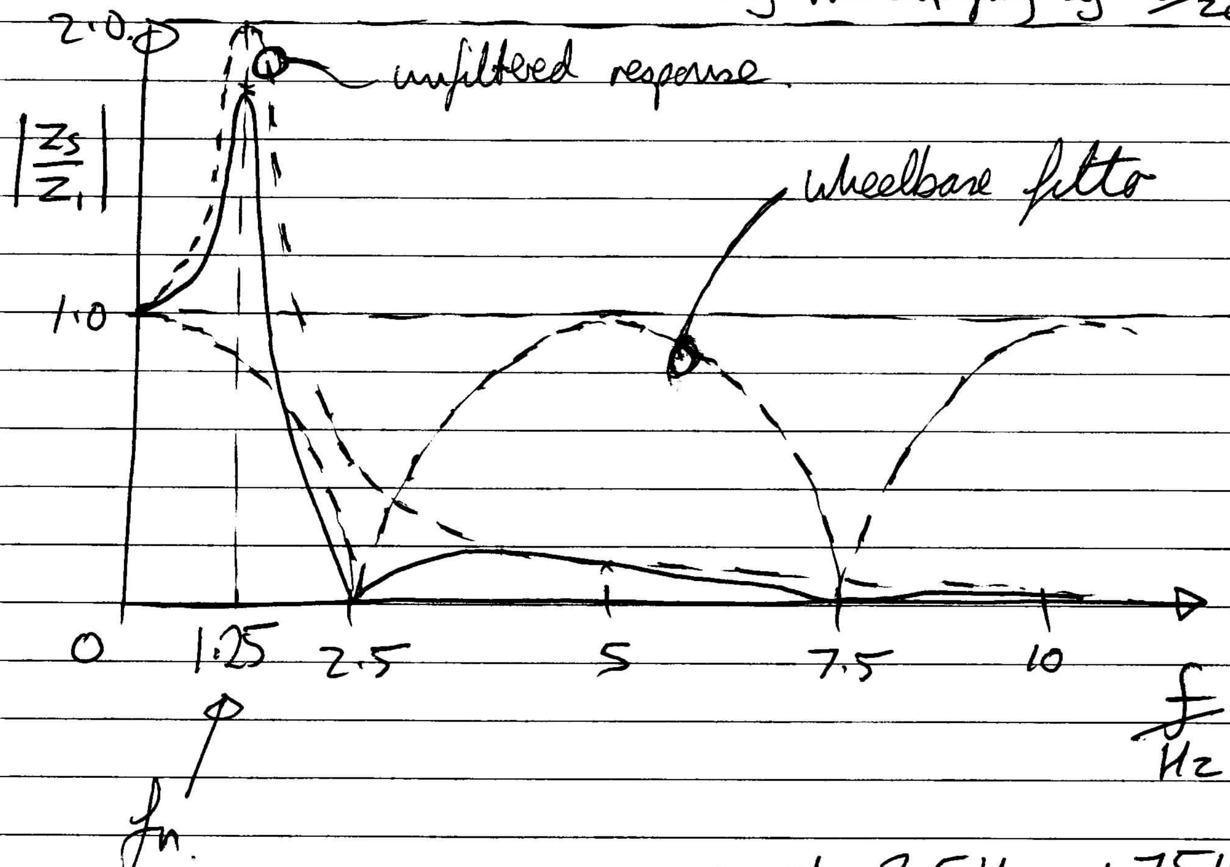
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{617.10^3}{10.10^3}}$$
$$= 7.854 \text{ rad/s}$$
$$f_n = \underline{\underline{1.25 \text{ Hz}}}$$

damping ratio $\zeta = \dfrac{c}{2\sqrt{mk}}$

$$= \frac{47.10^3}{2\sqrt{10.10^3 . 617.10^3}}$$

$$\zeta = \underline{\underline{0.3}}$$

convert spatial frequency $\frac{2a}{\lambda}$ to temporal frequency f by multiplying by $\frac{U}{2a}$



zero response at 2.5 Hz and 7.5 Hz.

c) let $z_1(t) = e^{j\omega t}$

(use temporal frequency for familiarity; could alternatively use spatial frequency)

$$z_2(t) = z_1(t)\, e^{-j\omega T} \qquad T = \frac{2a}{u}$$

$$= e^{j\omega t} e^{-j\omega T}$$

axle time delay

$$\therefore z_u(t) = \frac{z_1(t) + z_2(t)}{2}$$

$$= z_1(t)\left(\frac{1+e^{-j\omega T}}{2}\right)$$

$$|z_u| = \left|\frac{1+e^{-j\omega T}}{2}\right|$$

$$= \left|\frac{1+\cos(-\omega T) + j\sin(-\omega T)}{2}\right|$$

$$= \sqrt{\frac{(1+\cos(\omega T))^2 + (\sin(-\omega T))^2}{4}}$$

$$= \sqrt{\frac{1+2\cos\omega T + \cos^2\omega T + \sin^2\omega T}{4}}$$

$$= \sqrt{\frac{2(1+\cos\omega T)}{4}}$$

$$= \sqrt{\frac{1+\cos\omega T}{2}}$$

$$= \sqrt{\cos^2\frac{\omega T}{2}}$$

$$= \left|\cos\frac{\omega T}{2}\right|$$

$$|z_u| = \left|\cos \underbrace{\frac{u}{\lambda}.2\pi}_{\omega} . \underbrace{\frac{2a}{u}}_{T} . \frac{1}{2}\right| = \left|\cos\frac{2\pi a}{\lambda}\right|$$

2 a) $z_L = z_V - z_\phi \qquad z_R = z_V + z_\phi$

z_V and z_ϕ are uncorrelated so

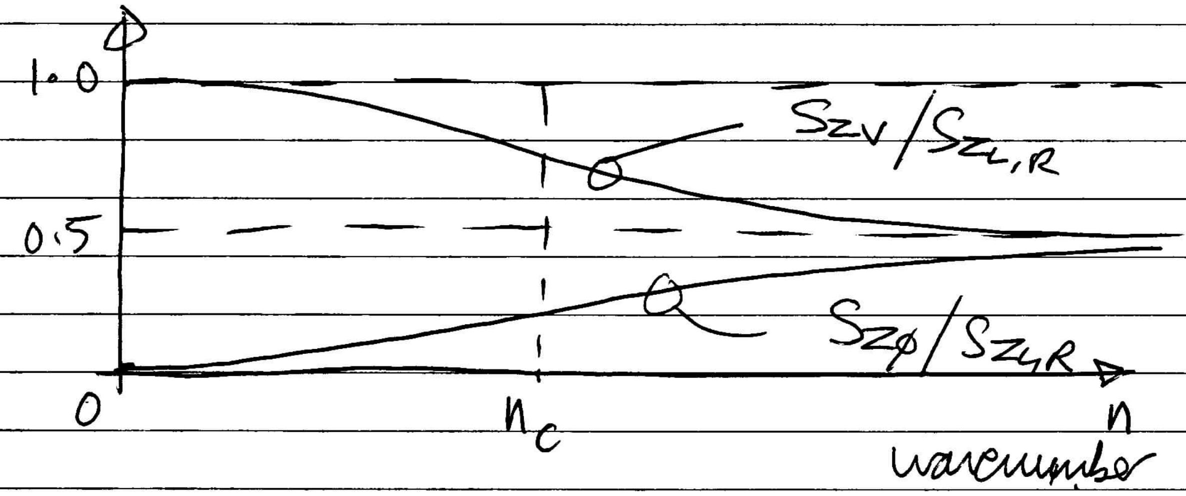
$$S_{z_L}(n) = S_{z_R}(n) = S_{z_V}(n) + S_{z_\phi}(n)$$

but $S_{z_\phi}(n) = |G(n)|^2 S_{z_V}(n)$

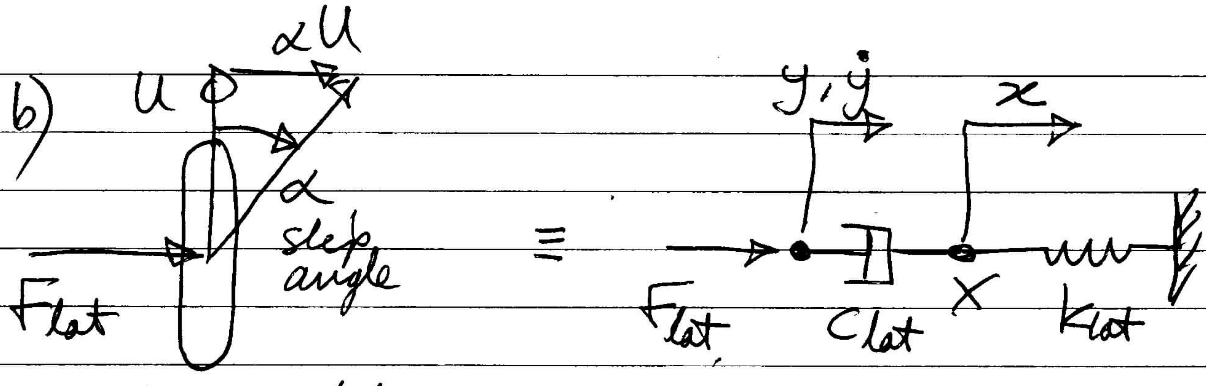
so $S_{z_{L,R}}(n) = S_{z_V}(n)\left(1 + |G(n)|^2\right)$

$$\therefore \quad \frac{S_{z_V}}{S_{z_{L,R}}} = \frac{1}{1+|G(n)|^2} = \frac{1}{1+\frac{n^2}{n_c^2+n^2}} = \frac{n_c^2+n^2}{n_c^2+2n^2}$$

and $$\frac{S_{z_\phi}}{S_{z_{L,R}}} = \frac{|G(n)|^2}{1+|G(n)|^2} = \frac{n_c^2+n^2}{n_c^2+2n^2}\cdot\frac{n^2}{n_c^2+n^2} = \frac{n^2}{n_c^2+2n^2}$$



b)



steady state
lateral tyre force $F_{lat} = \alpha . C$

equivalent viscous damping $c_{lat} = \dfrac{F_{lat}}{\alpha U}$

hence $c_{lat} = \dfrac{\alpha C}{\alpha U} = \dfrac{C}{U}$

equilibrium of point X

$$k_{lat} x = c_{lat}(\dot{y} - \dot{x})$$

$$k_{lat} x + c_{lat} \dot{x} = c \dot{y}$$

has solution $x = A e^{-\frac{k_{lat}}{c_{lat}} t}$

$$x = A e^{-t/\tau_{lat}}$$

where $\dfrac{c_{lat}}{k_{lat}} = \tau_{lat}$ is a time constant

substituting $c_{lat} = \dfrac{C}{U}$:

$$k_{lat} = \frac{C}{\tau_{lat} U} = \frac{C}{L_{relax}}$$

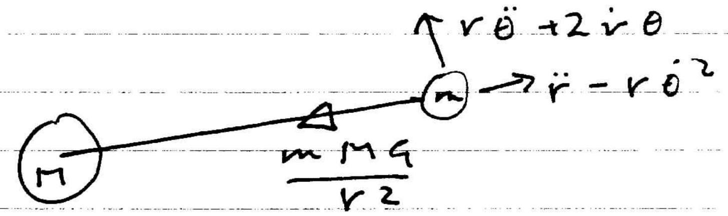
where L_{relax} is the relaxation length of the tyre.

c) At low speed the frequency corresponding to n_c is low and so the bounce and roll modes are both excited strongly. As speed increases the frequency corresponding to n_c increases and so the sprung mass roll modes receive less excitation than the sprung mass bounce modes.

The increase in speed also reduces C_{lat} leading to a change in the lateral-roll modes of vibration; the bounce modes are unaffected by the change in C_{lat}.

There is also an increase in excitation from the road due to the increase in vehicle speed.

3. a)



Radially, $m(r\dot{\theta}^2 - \ddot{r}) = \frac{mMG}{r^2}$

whence $r\dot{\theta}^2 - \ddot{r} = \mu/r^2$ where $\mu = MG$

Tangentially, $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$

$\therefore\ r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0$$

i.e. $r^2\dot{\theta}$ = constant = h, say

where h = specific angular momentum of orbiting mass.

b) From the above, $\dot{\theta} = h/r^2$

So $h^2/r^3 - \ddot{r} = \mu/r^2$

Letting $u = 1/r$,

$$\dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{d(1/u)}{d\theta} \times \frac{h}{r^2}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta} h u^2 = -h\frac{du}{d\theta}$$

$$\therefore\ \ddot{r} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2} \times hu^2 = -h^2u^2\frac{d^2u}{d\theta^2}$$

So $h^2u^3 + h^2u^2\frac{d^2u}{d\theta^2} = \mu u^2$

or $\dfrac{d^2u}{d\theta^2} + u = \mu/h^2$

c) General solution is $u = \mu/h^2 + A\cos(\theta - \theta_0)$

$\theta_0 = 0$ if θ is measured from periapsis, so we can write

$$u = \frac{\mu}{h^2}(1 + e\cos\theta)$$

whence $r = \dfrac{h^2/\mu}{1 + e\cos\theta}$

From the data sheet, $L = h^2/\mu$ and $a = \dfrac{L}{(1-e^2)}$, so $r = \dfrac{a(1-e^2)}{1 + e\cos\theta}$

a is the semi-major access of the ellipse, and e is its eccentricity.

d) For a given orbit, rate of sweeping is $\dfrac{r\dot{\theta}^2}{2} = h/2$ and is constant.

$\therefore$ Period $= \dfrac{\text{area of ellipse} \times 2}{h}$

From datasheet, $a = \dfrac{L}{(1-e^2)}$ and $b = \dfrac{L}{\sqrt{1-e^2}}$

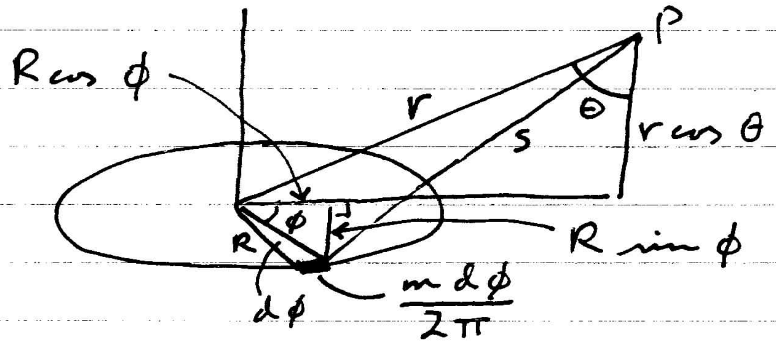
So $b^2 = aL$ and area $= \pi a b = \pi a^{3/2} L^{1/2}$

$\therefore$ Period $= \dfrac{2\pi a^{3/2} L^{1/2}}{h} = \dfrac{2\pi a^{3/2}}{\sqrt{\mu}}$

which is $\propto a^{3/2}$ and independent of h & e

4. a) J_2 comes from the earth's equatorial 'bulge', caused by the spin of the earth.

b)



Potential caused by mass shown is $\frac{mG\,d\phi}{2\pi s}$

But $s^2 = (R\sin\phi)^2 + (r\sin\theta - R\cos\phi)^2 + (r\cos\theta)^2$

$$= R^2\sin^2\phi + r^2\sin^2\theta + R^2\cos^2\phi - 2rR\sin\theta\cos\phi + r^2\cos^2\theta$$

$$\therefore\ s = \sqrt{R^2 + r^2 - 2Rr\sin\theta\cos\phi}$$

Hence total potential is $\frac{mG}{2\pi}\int_0^{2\pi} \frac{d\phi}{(R^2+r^2-2Rr\sin\theta\cos\phi)^{1/2}}$

c) If we let $\frac{R}{r} = a$ $(a \ll 1)$, the integral becomes $U = \frac{mG}{2\pi r}\int_0^{2\pi} (1 + a^2 - 2a\sin\theta\cos\phi)^{-\frac{1}{2}}\,d\phi$

Expanding ( ) with binomial theorem gives

$$\left(1 - \frac{a^2}{2} + a\sin\theta\cos\phi + \frac{3}{8}(a^2 - 2a\sin\theta\cos\phi)^2 + \dots\right)$$

Keeping terms up to a^2 gives

$$\left(1 - \frac{a^2}{2} + a\sin\theta\cos\phi + \frac{3}{2}a^2\sin^2\theta\cos^2\phi\right)$$

$$\text{So } U \approx \frac{mG}{2\pi r}\int_0^{2\pi} 1 - \frac{a^2}{2} + a \sin\theta\cos\phi + \frac{3a^2}{4}\sin^2\theta(1+\cos 2\phi)\, d\phi$$

$$= \frac{mG}{2\pi r}\left[\phi\left(1-\frac{a^2}{2}\right) + a\sin\theta\sin\phi + \frac{3a^2}{4}\sin^2\theta\left(\phi + \frac{\sin 2\phi}{2}\right)\right]_{\phi=0}^{\phi=2\pi}$$

$$\therefore \underline{U = \frac{mG}{r}\left(1 - \frac{R^2}{2r^2} + \frac{3}{4}\frac{R^2}{r^2}\sin^2\theta\right)}$$

d) Putting this into the 'standard form' gives

$$U = \frac{mG}{r}\left(1 - \frac{R^2}{2r^2}\left(1 - \frac{3}{2}\left(1-\cos^2\theta\right)\right)\right)$$

$$= \frac{mG}{r}\left(1 - \frac{R^2}{2r^2}\left(\frac{3\cos^2\theta - 1}{2}\right)\right)$$

$$= \frac{mG}{r}\left(1 - \frac{R^2}{2r^2}P_2(\cos\theta)\right)$$

Adding a spherical mass M at the origin gives

$$U' = \frac{(m+M)G}{r} - \frac{mG}{2r}\frac{R^2}{r^2}P_2(\cos\theta)$$

From data sheet,

$$U' = \frac{(m+M)G}{r} - \frac{(M+m)G}{r}\frac{R^2}{r^2}J_2 P_2(\cos\theta)$$

Equating these two gives

$$\frac{m}{2} = (M+m)J_2$$

$$\text{i.e. } \frac{m}{M+m} = 2J_2 = \underline{2.16\times 10^{-3}}$$

i.e. the fraction of the earth's mass that should be modelled into the ring.

ENGINEERING TRIPOS PART IIB 2016
ASSESSORS' COMMENTS, MODULE 4C8: APPLICATIONS OF DYNAMICS

Q1 Wheelbase filtering

Sketches of the filter function and explanations of wheelbase filtering in part (a) were generally good. Sketches of the transfer function in part (b) were often poorly annotated (incorrect or absent numerical values; missing axis labels or units), and the resonance of the vehicle mass on its suspension was sometimes not included. The derivation of the equation in part (c) was generally performed well.

Q2 Vehicle vibration in the roll plane

This question tested candidates' understanding of the effects of road surface and tyre properties on vehicle vibration in the roll plane. The majority of answers demonstrated good understanding.

Q3 Keplerian orbit

A relatively straightforward question, based on material covered in the lectures, which was very well answered by the great majority of candidates who attempted it. In the final part of the question, most candidates correctly identified the main causes for non-Kelperian behaviour, but only one reasoned that the magnitude of these effects would be smaller than the accuracy of Kepler's observations.

Q4 External potential of the earth

A reasonably challenging question, which was generally very well answered. Some candidates used an unnecessarily complex way of finding the distance to the ring (using the cosine rule instead of the sum of the squares), but most were then able to find the appropriate binomial approximation for the ring's potential, and hence answer the final part of the question correctly.

D J Cole (Principal Assessor)

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May 2016