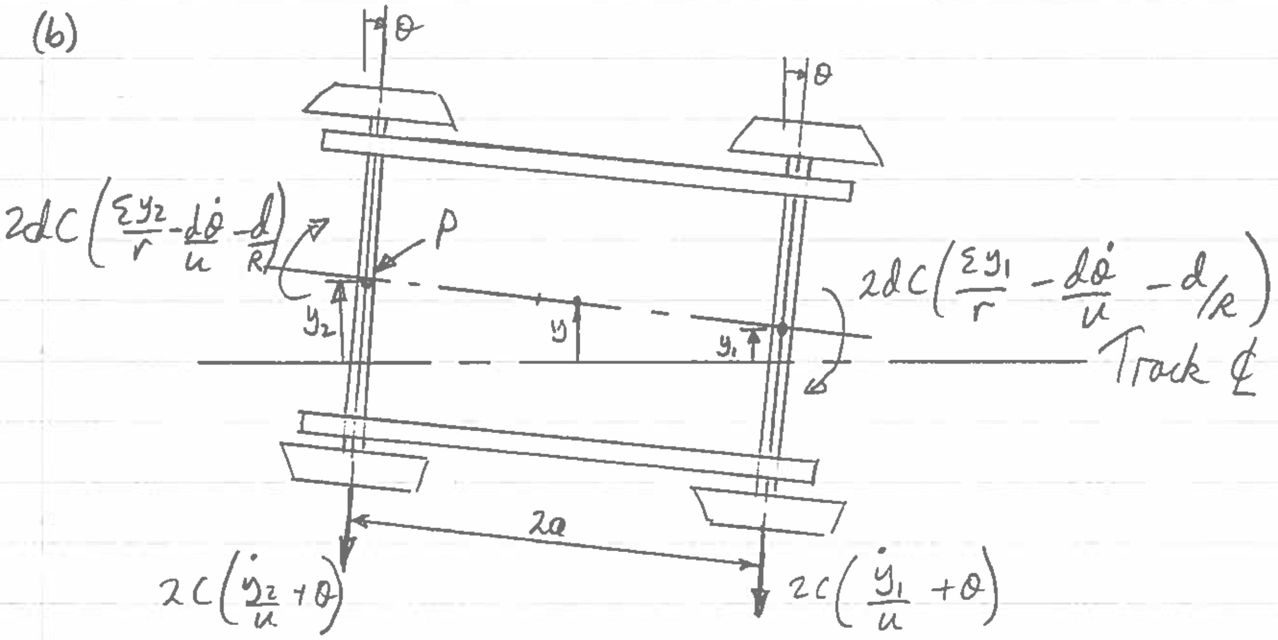
4C8 CRIB

PART IIB MODULE 4C8 VEHICLE DYNAMICS 2019

SOLUTIONS

1.(a) For derivation of force and moment, see lecture notes.

(b)



$$\Sigma M_P:\ \cancel{2}d\cancel{C}\left(\frac{\varepsilon y_1}{r}-\frac{2d\dot\theta}{u}-\frac{2d}{R}+\frac{\varepsilon y_2}{r}\right)+(2a)(\cancel{2C})\left(\frac{\dot y_1}{u}+\theta\right)=0 \qquad (1)$$

Now $y_1 = y - a\theta$ & $y_2 = y + a\theta$ (2)

$$\Sigma F = 0:\quad 2C\left(\frac{\dot y_1+\dot y_2}{u}+2\theta\right)=0$$

$$\Rightarrow \theta = -\frac{(\dot y_1+\dot y_2)}{2u} = -\frac{\dot y}{u} \qquad (3)$$

$$\& \quad \dot\theta = -\ddot y/u \qquad (4)$$

Combining (1) – (4) gives

$$\frac{\varepsilon}{r}\left[y-\cancel{a\theta}+y+\cancel{a\theta}\right]+\frac{\cancel{2}d}{u^2}\ddot y-\frac{\cancel{2}d}{R}+\frac{\cancel{2}a}{d}\left[\cancel{\frac{\dot y}{u}}+\frac{a\ddot y}{u^2}-\cancel{\frac{\dot y}{u}}\right]=0$$

ie $$\frac{1}{u^2}\left(d+\frac{a^2}{d}\right)\ddot y+\frac{\varepsilon}{r}y=\frac{d}{R}$$

ie $$\ddot y+\frac{\varepsilon u^2}{dr(1+a^2/d^2)}y=\left(\frac{u^2}{1+a^2/d^2}\right)\frac{1}{R} \qquad (5)$$

This is the motion of a forced, single D.O.F. undamped oscillator.

(c) If the track is straight then $R = \infty$ and RHS of (5) is zero. The undamped natural frequency is the coeff of the middle term:

$$\omega_n^2 = \frac{\varepsilon}{dr} \frac{u^2}{(1+a^2/d^2)} \qquad \text{—— (6)}$$

The kinematic hunting wavelength λ is:

$$\lambda = \frac{2\pi u}{\omega_n} = \underbrace{2\pi \sqrt{\frac{dr}{\varepsilon}}}_{\lambda_{\text{free wheelset}}} \sqrt{1 + \frac{a^2}{d^2}} \quad //$$

(d) Lateral track displacement $z = \Delta \sin \frac{2\pi x}{L}$

Curvature $\frac{1}{R} \approx \frac{d^2 z}{dx^2} = -\frac{4\pi^2}{L^2} \Delta \sin \frac{2\pi x}{L}$

Put $x = ut \Rightarrow \frac{1}{R} = -\frac{4\pi^2}{L^2} \Delta \sin \frac{2\pi u}{L} t \qquad \text{—— (7)}$

Write $\omega = \frac{2\pi u}{L}$ so $\frac{4\pi^2}{L^2} = \frac{\omega^2}{u^2} \qquad \text{—— (8)}$

(7) & (8) $\rightarrow \frac{1}{R} = -\frac{\omega^2}{u^2} \Delta \sin \omega t \qquad \text{—— (9)}$

So (5), (6) and (9) give

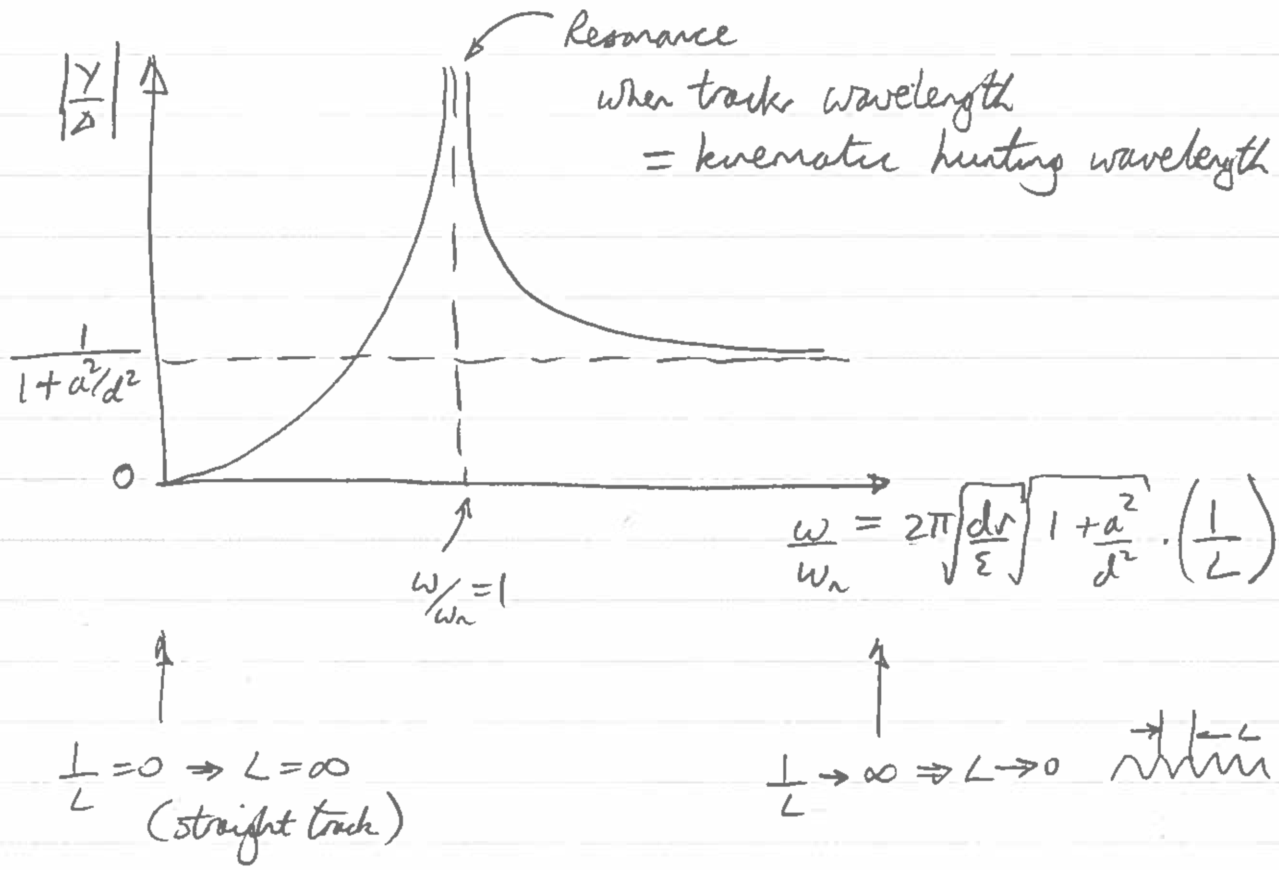
$$\ddot{y} + \omega_n^2 y = \frac{-\cancel{d}}{(1+a^2/d^2)} \frac{\omega^2}{\cancel{u^2}} \Delta \sin \omega t \qquad \text{—— (10)}$$

Replacing $\sin \omega t$ with $e^{i\omega t}$ and putting $y = Y e^{i\omega t}$ gives the forced vibration solution:

$$(-\omega^2 + \omega_n^2) Y = \frac{\omega^2}{1+a^2/d^2} \Delta$$

$$\Rightarrow \frac{Y}{\Delta} = \frac{-1}{1+a^2/d^2} \frac{\omega^2}{\omega_n^2 - \omega^2}$$

ie $$\left| \frac{Y}{\Delta} \right| = \left(\frac{1}{1+a^2/d^2} \right) \frac{\omega^2/\omega_n^2}{\left| 1 - \omega^2/\omega_n^2 \right|}$$



$\frac{1}{L} = 0 \rightarrow$ For straight track, $\left|\frac{Y}{\Delta}\right| \to 0$ as expected since the complementary function of ⑤ dies out for the smallest amount of damping.

2 a) κ is the roughness, numerically equal to the spectral density at a wavenumber $n = 1$ cycle/m. A larger value means a rougher road surface.

n is wavenumber, the number of wavelengths per unit distance along the road, or the inverse of wavelength.

w is the downwards gradient of the spectrum on log axes. Most road surfaces have $w \simeq 2.5$.

Single-sided means that the mean square displacement is evaluated over positive wavenumbers only.

b) From definitions in (a) and from Figure, deduce that

$$w = 2.5$$

$$\kappa = 10^{-6} \ m^{\frac{1}{2}} \text{cycle}^{\frac{3}{2}}$$

c) i) $S_{z_r}(n) = \kappa n^{-w}$

but $\omega = n 2\pi U$ (note distinction between w and ω)

so $S_{z_r}(n) = \kappa \left(\dfrac{\omega}{2\pi U}\right)^{-w}$

Must ensure that $\displaystyle\int_{n=0}^{n=\infty} S_{z_r}(n)\,dn = \int_{\omega=0}^{\omega=\infty} S_{z_r}(\omega)\,d\omega = E[z_r^2]$

Now $d\omega = 2\pi U\,dn$

so $\displaystyle\int_{\omega=0}^{\omega=\infty} \kappa\left(\frac{\omega}{2\pi U}\right)^{-w}\frac{d\omega}{2\pi U} = \int_{\omega=0}^{\omega=\infty} S_{z_r}(\omega)\,d\omega$

therefore $S_{z_r}(\omega) = \dfrac{\kappa}{2\pi U}\left(\dfrac{\omega}{2\pi U}\right)^{-w}$

$$S_{z_r}(\omega) = \kappa(2\pi U)^{w-1}\omega^{-w}$$

ii) Assume that $w = 2$ (instead of 2.5)
so that $S_{z_r}(n) = \kappa n^{-2}$

from (i) $S_{z_r}(\omega) = \kappa 2\pi U \omega^{-2}$

multiply by ω^2 to give white noise spectrum on RHS:

$$\omega^2 S_{z_r}(\omega) = \kappa 2\pi U$$

LHS is spectral density of $\dfrac{dz_r}{dt}$ i.e. velocity of road input

RHS is singled-sided spectral density of white noise, but S_0 is defined for double sided spectrum

$\therefore S_0 = \kappa \pi U$

iii) From the result in part (c)(ii) the input to $G(j\omega)$ must be $\dot{Z}_r(j\omega)$,
which is $(j\omega) Z_r(j\omega)$

Hence require $\underline{\underline{G(j\omega) = \left(\frac{1}{j\omega}\right) \cdot H(j\omega)}}$

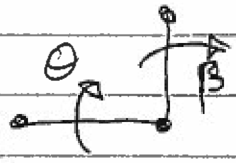
$$Z_r(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow \ddot{Z}_s(j\omega)$$

$$\dot{Z}_r(j\omega) \longrightarrow \underbrace{\boxed{\frac{1}{j\omega}} \xrightarrow{Z_r(j\omega)} \boxed{H(j\omega)}}_{G(j\omega)} \longrightarrow \ddot{Z}_s(j\omega).$$

3

a) Newton's 2nd law about A

let $\theta = \frac{z_2 - z_1}{l}$ and $\beta = \frac{y}{h}$



$$(mh^2)\ddot{\beta} = k(\theta - \beta) + c(\dot{\theta} - \dot{\beta})$$

Laplace $mh^2 s^2 \beta = k(\theta - \beta) + sc(\theta - \beta)$

$$= (k + cs)\,\theta - (k + cs)\,\beta$$

$$mh^2 s^2 \frac{y}{h} = (k + cs)\left(\frac{z_2 - z_1}{l}\right) - (k + cs)\frac{y}{h}$$

$$(mh^2 s^2 + cs + k)\,y = (k + cs)\,\frac{h}{l}\,(z_2 - z_1)$$

let $s = j\omega$

$$\left(mh^2 (j\omega)^2 + c j\omega + k\right) y(j\omega) = (k + c j\omega)\frac{h}{l}\left(z_2(j\omega) - z_1(j\omega)\right)$$

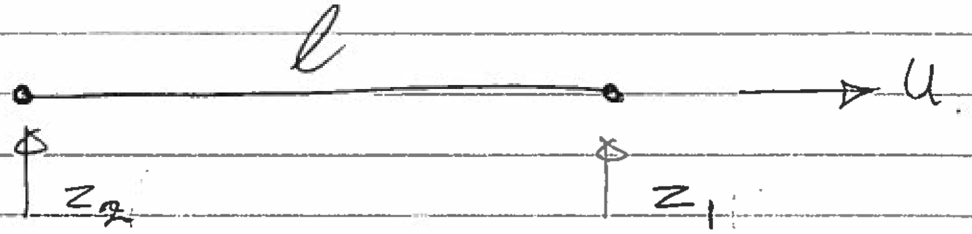
b) divide through by k and compare to D.B. case (c)

$$\underbrace{\left(\frac{mh^2}{k}(j\omega)^2 + \frac{c}{k} j\omega + 1\right)}\, y(j\omega) = \left(\frac{c}{k} j\omega + 1\right) \underbrace{\frac{h}{l}\left(z_2(j\omega) - z_1(j\omega)\right)}_{\text{input}}$$

$$\frac{mh^2}{k} = \frac{1}{\omega_n^2}$$

$$\therefore \omega_n = \sqrt{\frac{k}{mh^2}}$$

c)



$$z_2(t) = z_1\left(t - \frac{l}{u}\right)$$

Laplace transform and $s = j\omega$

$$z_2(j\omega) = z_1(j\omega)\, e^{-j\omega \frac{l}{u}}$$

$$\therefore\ z_2(j\omega) - z_1(j\omega) = z_1(j\omega)\left[e^{-j\omega\frac{l}{u}} - 1\right]$$

$z_2(j\omega) - z_1(j\omega)$ is zero when

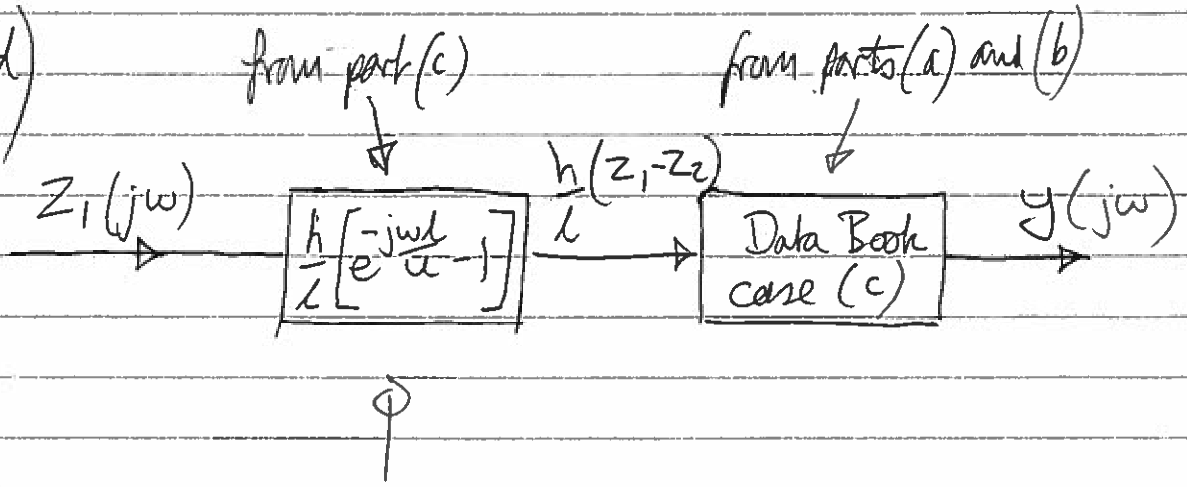
$$e^{-j\omega\frac{l}{u}} = 1$$

$$\therefore\ \frac{\omega l}{u} = N 2\pi$$

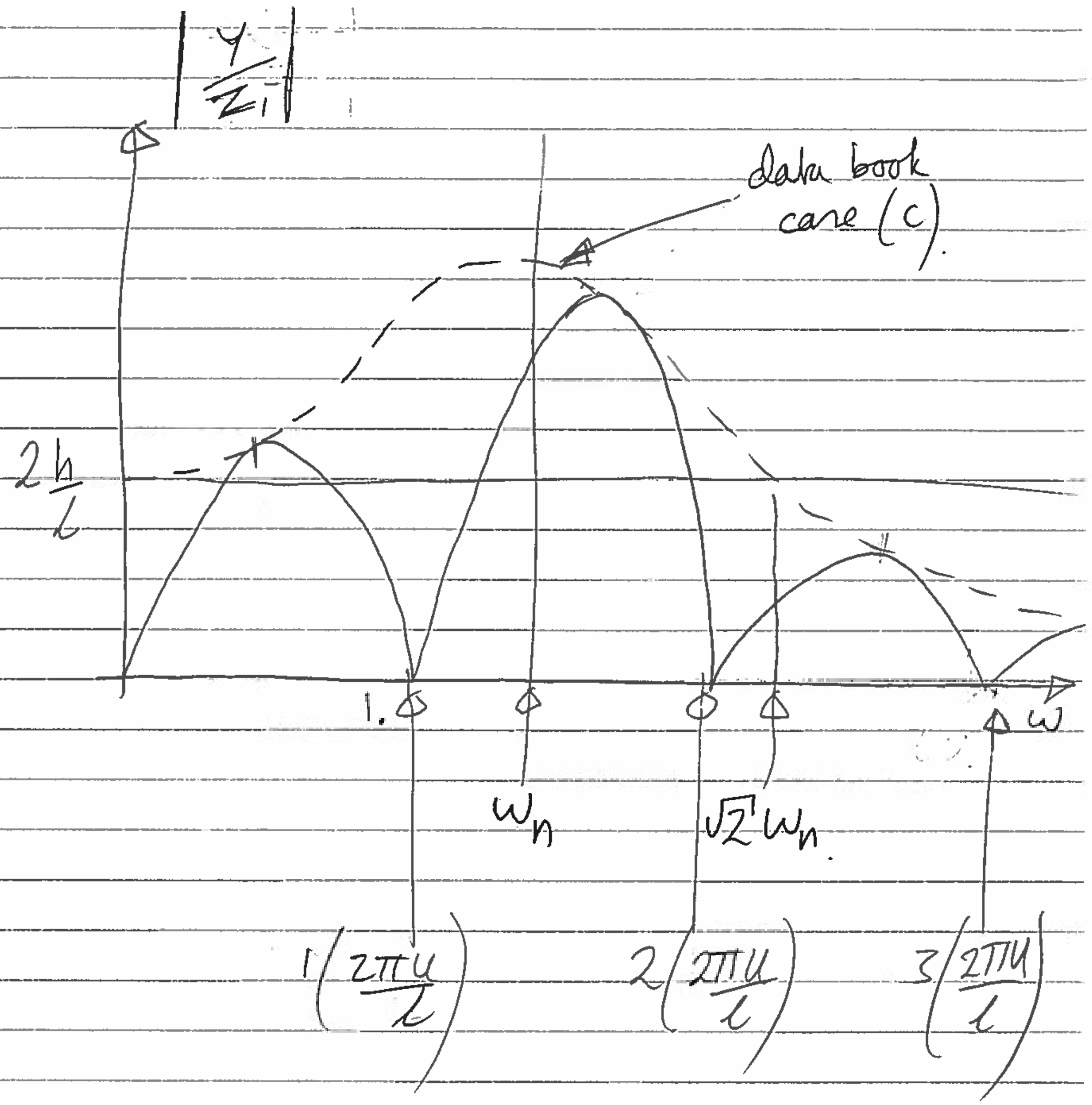
(N is integer)

$$\omega = N\, 2\pi \frac{u}{l}$$

d)



zero when $\omega = \frac{N 2\pi u}{l}$; max value is $\frac{2h}{l}$.



Reduce discomfort by setting first trough of wheelbase filter equal to the natural frequency,

$$1.\left(\frac{2\pi u}{l}\right) = \omega_n = \sqrt{\frac{k}{mh^2}}$$

$$\therefore\ k = \frac{4\pi^2 u^2 m h^2}{L^2}$$

or set $k = 0$

4(a) Bookwork - see lecture notes

(b) $$m(\dot{v} + u\Omega) + \frac{2cv}{u} + (a-b)\frac{c\Omega}{u} = c\delta$$

$$I\dot{\Omega} + c(a-b)\frac{v}{u} + c(a^2+b^2)\frac{\Omega}{u} = ac\delta$$

Put $\delta = -K\Omega$ and write in matrix form:

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} \dot{v} \\ \dot{\Omega} \end{Bmatrix} + \begin{bmatrix} 2c/u & mu + (a-b)c/u + Kc \\ c(a-b)/u & c(a^2+b^2)/u + aKc \end{bmatrix}\begin{Bmatrix} v \\ \Omega \end{Bmatrix} = \underset{\sim}{0}$$

for stability put $v = \bar{v}e^{\lambda t}$ & $\Omega = \bar{\Omega}e^{\lambda t}$ and solve the characteristic equation:

$$\begin{vmatrix} mu\lambda + 2c & mu^2 + (a-b)c + Kcu \\ c(a-b) & Iu\lambda + c(a^2+b^2) + aKcu \end{vmatrix} = 0$$

$$\Rightarrow (mu\lambda + 2c)(Iu\lambda + c(a^2+b^2) + aKcu) - c(a-b)(mu^2 + (a-b)c + Kcu) = 0$$

$$\Rightarrow \underbrace{(Imu^2)}_{a_2}\lambda^2 + \underbrace{\{2cIu + mu[c(a^2+b^2) + aKcu]\}}_{a_1}\lambda$$

$$+ \underbrace{\{2c[c(a^2+b^2) + aKcu] - c(a-b)(mu^2 + (a-b)c + Kcu)\}}_{a_0} = 0$$

(c) Stability conditions: All a's > 0

(i) $a_2 > 0$ always

(ii) $a_1 > 0$ if $K > 0$

If $K < 0$, then there the vehicle is stable for

$\cancel{c}(a^2+b^2) - a|K|\cancel{c}u > 0$ ie $u < \dfrac{(a^2+b^2)}{a|K|}$

(iii) $a_3 > 0$:

$$2c(a^2+b^2) + 2aKcu - (a-b)mu^2 - (a-b)^2c - (a-b)Kcu > 0$$

ie $(b-a)mu^2 + (a+b)Kcu + (a+b)^2c > 0$ ——— ①

If $b > a$ & $K \geqslant 0$, this is always stable

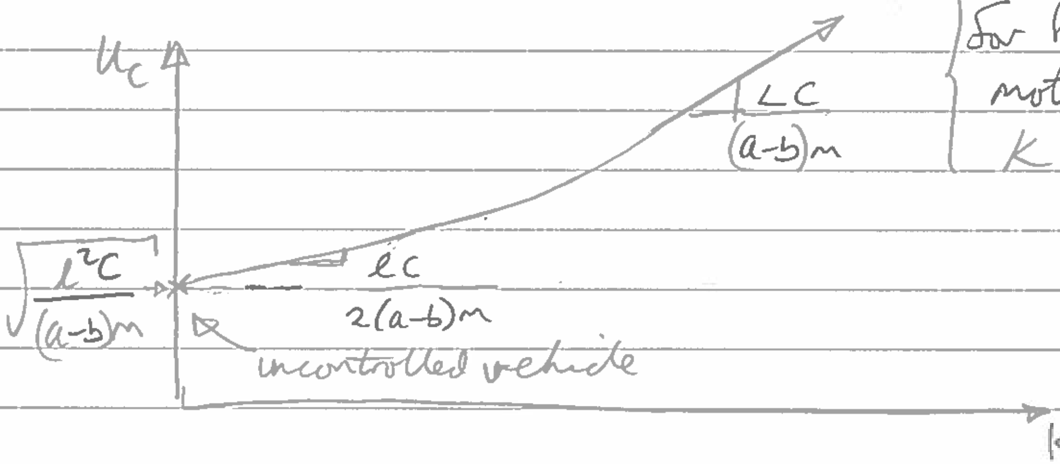
For $a > b$ & $K \geq 0$, critical speeds can be found from the solutions to (1) with $l = a + b$:

$$u_c = \frac{-lKC \pm \sqrt{(lKC)^2 - 4(b-a)m\,l^2C}}{2(b-a)m}$$

$$= \frac{lKC}{2(a-b)m} \pm \sqrt{\frac{(lKC)^2 + 4(a-b)m\,l^2C}{4(a-b)^2m^2}}$$

for $K = 0$, $u_c = \sqrt{\frac{l^2C}{(a-b)m}}$ ✓ as per lecture notes for uncontrolled vehicle.

For $K \to \infty$ and +ve root, $u_c \to \frac{lC}{(a-b)m} K$



For forwards motion, increasing K increases u_c

4C8: Examiner's comments:

Question 1 Attempts: 28; average raw mark: 12.1; maximum:20; minimum 5.
Railway Bogie Forced Hunting. Part (a) was bookwork and generally fine. Part (b)(i) was relatively poorly done. (b)(ii) was fine. Only a few candidates got to (b)(iii).. most had trouble converting the sinusoidal displacement into curvature.

Question 2 Attempts: 26; average raw mark: 13.1; maximum:20; minimum 4.
Spectral density of road profile displacement. In part (a) many candidates simply stated the name of each parameter, rather provide an explanation. The value of w in part (b) was often inaccurately estimated as 2, whereas the correct value is 2.5. In part (c)(ii) account was usually not taken of Fig.3 being single-sided and S_0 being double-sided. The expression derived for part (c)(iii) often omitted any phase information.

Question 3 Attempts: 18; average raw mark: 9.5; maximum:18; minimum 2.
Wheelbase filter and pitch-plane vibration. Most solutions to part (a) were incorrect to some extent; there was confusion about the relationship of translational to rotational displacements, and about the relationship of forces to moments. In part (b), discrepancies with the Data Book expression usually did not prompt candidates to review their answer to part (a). There were some good answers to part (d), but there were also many instances of messy sketches, insufficiently annotated.

Question 4 Attempts: 36; average raw mark: 14.2; maximum: 19; minimum 7.
Autonomous car steering: Part (a) was bookwork and well done. Part (b) had a bi-modal mark distribution. Most did reasonably well, the remainder flopped for no consistent reason. The overall average was high.