4C9 Solutions

2015/16

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Q1

(a) (i) $(\underline{a} \otimes \underline{b}) \cdot \underline{c} = a_i b_j c_j$

$= a_j b_i c_j$ if symmetric : $a_i b_j = a_j b_i$

$= \underline{c} \cdot (\underline{a} \otimes \underline{b})$ [10%]

(ii) $\nabla \times (\nabla \times \underline{a}) = e_{ijk} \frac{\partial}{\partial x_j}\left(e_{kpq} \frac{\partial}{\partial x_p} a_q\right)$

$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_p} a_q$

$= \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} a_j - \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_p} a_i$

$= \nabla(\nabla \cdot \underline{a}) - (\nabla \cdot \nabla)\underline{a}$ [20%]

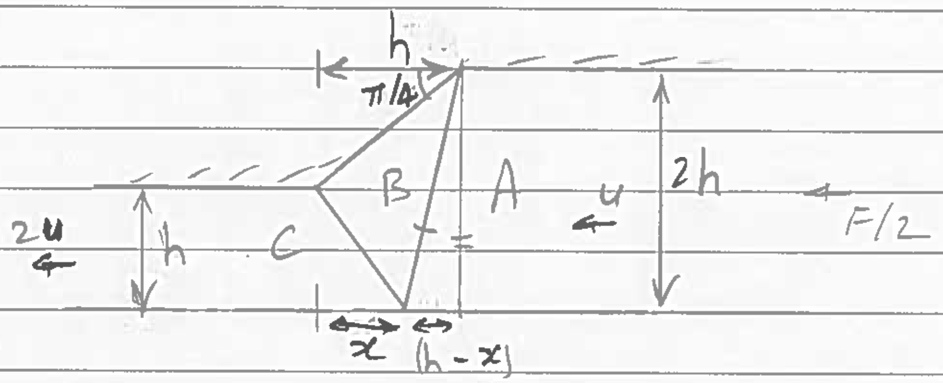
(iii) $\oint_S (\underline{a} \otimes \underline{b}) \cdot \underline{n} \, dS = \oint_S a_i b_j n_j \, dS$

$= \int_V \frac{\partial}{\partial x_j}(a_i b_j) \, dV$ (divergence theorem)

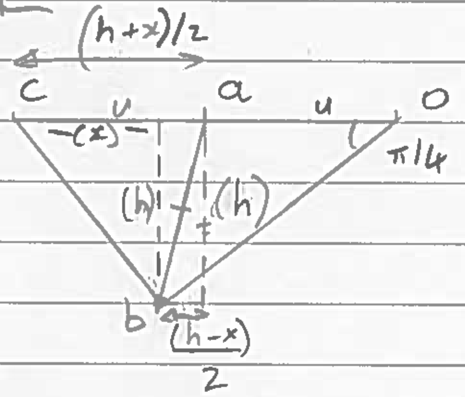
$= \int_V \left[\frac{\partial a_i}{\partial x_j} b_j + a_i \frac{\partial b_j}{\partial x_j}\right] dV$

$= \int_V \left[\underline{b} \cdot \nabla \underline{a} + \underline{a} (\nabla \cdot \underline{b})\right] dV$ [20%]

1. (b) (i)



Hodograph



choose scale of velocity diagram so that the vertical velocity of point b is proportional to h.

$$\frac{F}{2} u = k\left(l_{AB} v_{AB} + l_{BC} v_{BC} \right)$$

Now, $l_{AB}^2 = 4h^2 + (h-x)^2$

$$l_{BC}^2 = h^2 + x^2$$

$$\frac{v_{AB}}{u} = \frac{\sqrt{h^2 + \frac{1}{4}(h-x)^2}}{\frac{1}{2}(h+x)}$$

$$\frac{v_{bc}}{u} = \frac{\sqrt{h^2 + x^2}}{\frac{1}{2}(h+x)}$$

$$\frac{Fu}{2k} = u\,\frac{\sqrt{4h^2+(h-x)^2}\;\sqrt{h^2+\frac{1}{4}(h-x)^2}}{\frac{1}{2}(h+x)}$$

$$+ \quad u \quad \frac{\sqrt{h^2+x^2}\;\;\sqrt{h^2+x^2}}{\frac{1}{2}(h+x)}$$

$$\Rightarrow \quad \frac{F}{k} = \frac{4h^2+(h-x)^2 \; + \; 2h^2+2x^2}{\frac{1}{2}(h+x)}$$

$$\Rightarrow \quad \frac{F}{2k} = \frac{6h^2+(h-x)^2+2x^2}{h+x} = \frac{6h^2+h^2-2hx+3x^2}{h+x}$$

$$\frac{F}{2k} = \frac{7h^2-2hx+3x^2}{h+x}$$

$$\Rightarrow \quad \frac{F}{2kh} = \frac{7-2\xi+3\xi^2}{1+\xi} \qquad \xi = x/h$$

$$\frac{\partial(F/2kh)}{\partial\xi} = 0 \quad \Rightarrow \quad (1+\xi)(-2+6\xi) = 7-2\xi+3\xi^2$$

$$\Rightarrow \quad -2+6\xi-\cancel{2\xi}+6\xi^2 = 7-\cancel{2\xi}+3\xi^2$$

$$\Rightarrow \quad 3\xi^2+6\xi-9=0$$

$$\xi + 2\xi - 3 = 0$$

$$(\xi+1)^2 = 5 \quad \Rightarrow \quad \xi = -1+\sqrt{5}$$

So $\quad x_{opt} = h.(-1+\sqrt{3})$.

(ii) An assumed velocity field ⇒ upper bound.
slip lines are orthogonal in the exact slip-line field solution.

(EXAMINER: popular, upper bound understood, algebra was challenging)

Q2

(a)(i) Relaxation response:

Apply $\sigma(t) = \sigma_0 H(t) \longrightarrow \varepsilon(t) = \sigma_0 J_c(t)$

For $\sigma = \sigma_0$ (constant):

$$\frac{\eta}{E}\frac{d\varepsilon}{dt} + \varepsilon = \frac{\sigma_0}{E}$$

$$\int_0^{\varepsilon} \frac{d\varepsilon}{\frac{\sigma_0}{E} - \varepsilon} = \frac{E}{\eta}\int_0^t dt$$

$$-\left[\ln\left(\frac{\sigma_0}{E} - \varepsilon\right)\right]_0^{\varepsilon} = \frac{Et}{\eta}$$

$$\ln\left(\frac{\frac{\sigma_0}{E} - \varepsilon}{\sigma_0/E}\right) = -\frac{Et}{\eta}$$

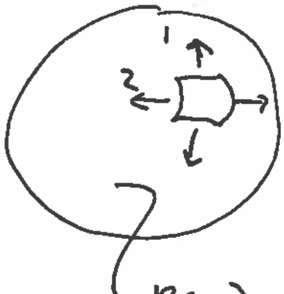
$$\therefore \; \varepsilon = \frac{\sigma_0}{E}\left(1 - e^{-\frac{Et}{\eta}}\right)$$

$$\therefore \; J_c(t) = \frac{\varepsilon(t)}{\sigma_0} = \frac{1}{E}\left(1 - e^{-\frac{Et}{\eta}}\right)$$

$$\Rightarrow A = \frac{1}{E}, \; B = \frac{-E}{\eta}$$

[20%]

(ii) Sphere:



P(t)

$$\sigma_{11} = \sigma_{22} = \frac{P(t)R}{2h}$$

(equilibrium)

$\sigma_{33} = 0$ (thin walled)

Constitutive equations:

$$\varepsilon_{11} = (1+\nu)\int_0^t J_c(t-\tau)\,\frac{\partial\sigma_{11}(\tau)}{\partial\tau}\,d\tau$$

$$-\nu\int_0^t J_c(t-\tau)\Big[\frac{\partial\sigma_{11}(\tau)}{\partial\tau} + \frac{\partial\sigma_{22}(\tau)}{\partial\tau} + \cancel{\frac{\partial\sigma_{33}(\tau)}{\partial\tau}}\Big]\,d\tau$$

$\sigma_{22}(\tau) = \sigma_{11}(\tau)$, $\quad \sigma_{33} \to 0$

$$= (1-\nu)\int_0^t J_c(t-\tau)\,\frac{\partial\sigma_{11}(\tau)}{\partial\tau}\,d\tau$$

$$= (1-\nu)\int_0^t \frac{1}{E}\Big(1-e^{-\frac{E(t-\tau)}{\eta}}\Big)\frac{\dot{p}R}{2h}\,d\tau$$

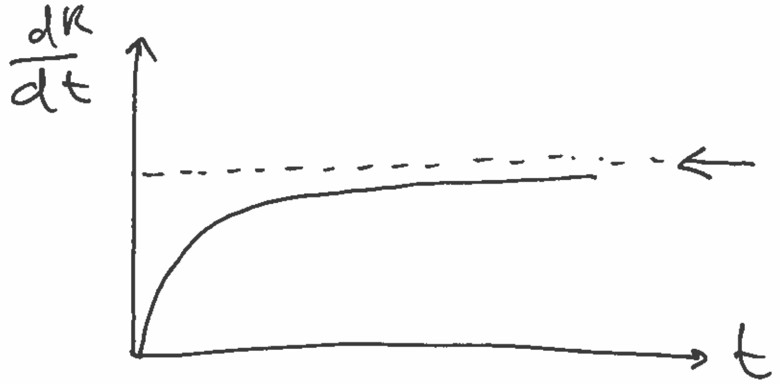
$$= \frac{\dot{p}R}{2h}\frac{(1-\nu)}{E}\Big[\tau - \frac{\eta}{E}e^{-\frac{E}{\eta}(t-\tau)}\Big]_{\tau=0}^{\tau=t}$$

$$= \frac{\dot{p}R}{2h}\frac{(1-\nu)}{E}\Big[t - \frac{\eta}{E} + \frac{\eta}{E}e^{-\frac{Et}{\eta}}\Big]$$

Change in radius:

$$\varepsilon_{11} = \frac{dR}{R} \qquad \therefore \frac{d\varepsilon_{11}}{dt} = \frac{1}{R}\frac{dR}{dt}$$

$$\therefore \frac{dR}{dt} = \frac{\dot{p}R^2(1-\nu)}{2hE}\Big[1-e^{-\frac{Et}{\eta}}\Big]$$



$\leftarrow \dfrac{\dot{p}R^2(1-\nu)}{2hE}$ = elastic solution (i.e. $\eta \to 0$)

[30%]

(6)

2 (b) (i) Drucker stability

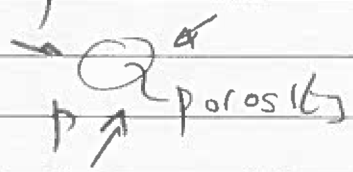
$$\oint \Delta\sigma_{ij}\, d\varepsilon_{ij} \geq 0$$

additional loading

Hence convexity & normality.

(ii) A fully dense metal yields by dislocation motion — a slip process, with no change of volume.

A porous solid can compact under pressure, by plastic flow around the voids.

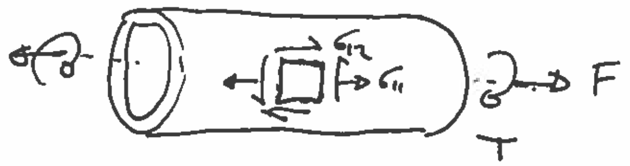


Hence, the volume can change.

Examiner: Many used Laplace transforms for visco-elastic part. No need to derive spherical expansion solution for $J_c(t)$ again.

Q3

(a) (i)



Non-zero stress components : σ_{11}, σ_{12}

Constitutive equations : $\varepsilon_{11} = \frac{\sigma_{11}}{E}$, $\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} = \frac{\gamma}{2}$

Elastic strain energy per unit volume:

$$U = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = \tfrac{1}{2}E\varepsilon_{11}^2 + 2\,\tfrac{1}{2}\,\frac{E}{1+\nu}\,\varepsilon_{12}^2$$

$$= \tfrac{1}{2}E\varepsilon_{11}^2 + \tfrac{1}{2}\,\frac{E}{2(1+\nu)}\,\gamma^2$$

Total elastic strain energy:

$$W_{int} = \int U\,dV = 2\pi R h L\left[\tfrac{1}{2}E\varepsilon_{11}^2 + \tfrac{1}{2}\,\frac{E}{2(1+\nu)}\,\gamma^2\right]$$

[15%]

(ii) End rotation: $\theta = \frac{L}{R}\gamma$

End displacement: $\Delta = \varepsilon_{11}L$

Variation in potential energy:

$$\delta\Pi = \int \delta U\,dV - T\delta\theta - F\delta\Delta$$

$$\delta U = \frac{\partial U}{\partial \varepsilon_{11}}\delta\varepsilon_{11} + \frac{\partial U}{\partial \gamma}\delta\gamma$$

$$= E\varepsilon_{11}\delta\varepsilon_{11} + \frac{E}{2(1+\nu)}\gamma\,\delta\gamma$$

$$\delta\theta = \frac{L}{R}\delta\gamma, \quad \delta\Delta = L\,\delta\varepsilon_{11}$$

(ii) cont.

At equilibrium: $\delta\Pi = 0$

$$\therefore\ 2\pi R h L\left[E\varepsilon_{11}\,\delta\varepsilon_{11} + \frac{E}{2(1+\nu)}\gamma\,\delta\gamma\right] - T\frac{L}{R}\delta\gamma - FL\,\delta\varepsilon_{11} = 0$$

Rearranging (group terms):

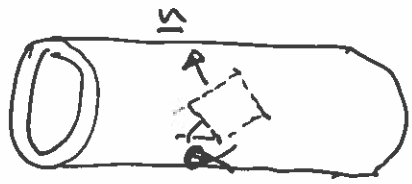
$$\left[2\pi R h L E\varepsilon_{11} - FL\right]\delta\varepsilon_{11} + \left[2\pi R h L\frac{E}{2(1+\nu)}\gamma - T\frac{L}{R}\right]\delta\gamma = 0$$

To hold for any $\delta\varepsilon_{11}$, $\delta\gamma$:

$$\varepsilon_{11} = \underbrace{\frac{F}{2\pi R h E}}_{\frac{\sigma_{11}}{E}}\ , \qquad \gamma = \underbrace{\frac{\left(\frac{T}{2\pi R^2 h}\right)}{\frac{E}{2(1+\nu)}}}_{\frac{\sigma_{12}}{\left[\frac{E}{2(1+\nu)}\right]}}$$

[20%]

(iii)



$\underline{n} = \cos\alpha\,\underline{e}_1 + \sin\alpha\,\underline{e}_2$

Traction: $t_i = \sigma_{ij} n_j$

Stress components: $\sigma_{11} = \dfrac{F}{2\pi R h}$, $\sigma_{12} = \sigma_{21} = \dfrac{T}{2\pi R^2 h}$

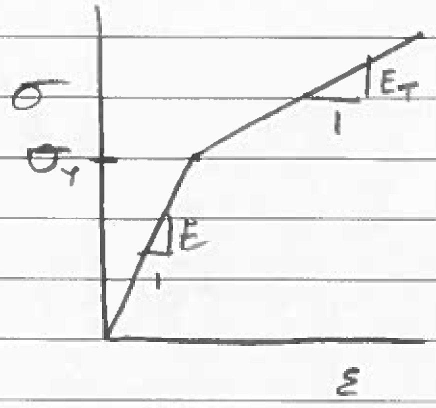
$$t_1 = \sigma_{12} n_2 + \sigma_{11} n_1$$

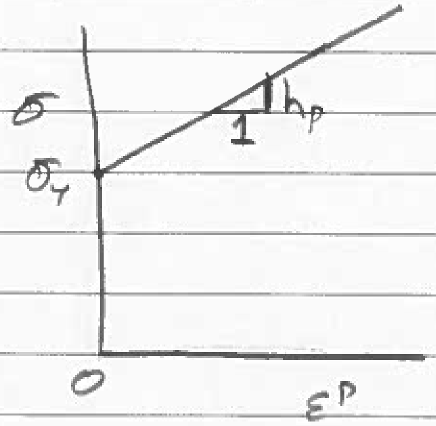
$$= \frac{T\sin\alpha}{2\pi R^2 h} + \frac{F\cos\alpha}{2\pi R h}$$

$$t_2 = \sigma_{21} n_1 = \frac{T\cos\alpha}{2\pi R^2 h}$$

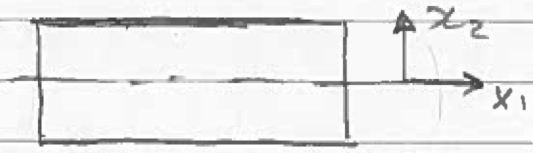
[15%]

3. (b)





$\sigma_e^2 = \frac{3}{2} s_{ij} s_{ij}$ $\qquad T = \alpha R F$



$F = 2\pi R h \, \sigma_{11}$ $\qquad \sigma_{22} = \sigma_{33} = 0$

$T = 2\pi R^2 h \, \sigma_{12}$

$T = \alpha R F \quad \Rightarrow \quad \underline{\sigma_{12} = \alpha \sigma_{11}}$

$\sigma_h = \frac{1}{3}\sigma_{11} \quad \Rightarrow \quad s_{11} = \frac{2}{3}\sigma_{11}$

$s_{22} = s_{33} = -\frac{1}{3}\sigma_{11}$

$$\sigma_e^2 = \frac{3}{2}\left(\left(\frac{2}{3}\right)^2 \sigma_{11}^2 + \left(\frac{1}{3}\right)^2 \sigma_{11}^2 + \left(\frac{1}{3}\right)^2 \sigma_{11}^2 + 2\sigma_{12}^2\right)$$

$$\sigma_e^2 = \sigma_{11}^2 + 3\sigma_{12}^2$$

(i)

~~At~~ At onset of yield : $\sigma_e = \sigma_Y$

$\Rightarrow \left(1 + 3\alpha^2\right)\sigma_{11}^2 = \sigma_Y^2$

$$\Rightarrow \left|\sigma_{11}\right|_Y = \frac{\sigma_Y}{\sqrt{1+3\alpha^2}} \qquad \left|\sigma_{12}\right|_Y = \frac{\alpha \sigma_Y}{\sqrt{1+3\alpha^2}}$$

3(b) (i) contd.

$$T_Y = 2\pi R^2 h \left(\sigma_{12}\right)_Y$$

$$\Rightarrow \quad |T_Y| = \frac{2\pi\alpha R^2 h\ \sigma_Y}{\sqrt{1+3\alpha^2}}$$

(ii) $$\dot{\varepsilon}_{ij}^{PL} = \frac{3}{2}\frac{S_{ij}}{\sigma_e}\frac{\dot{\sigma}_e}{h} \qquad \frac{1}{h_p} = \frac{1}{E_T} - \frac{1}{E} = \frac{E-E_T}{EE_T}$$

$$\dot{\varepsilon}_e^2 = \frac{2}{3}\dot{\varepsilon}_{ij}^{PL}\dot{\varepsilon}_{ij}^{PL} = \left(\frac{\dot{\sigma}_e}{h_p}\right)^2$$

Now, $$\sigma_{11} = \frac{F}{2\pi R h} \qquad \sigma_{12} = \frac{T}{2\pi R^2 h} = \alpha\,\sigma_{11}$$

$$\sigma_e^2 = \sigma_{11}^2 + 3\sigma_{12}^2 = \left(\frac{\sigma_{12}}{\alpha}\right)^2 + 3\sigma_{12}^2$$

$$= \left(\frac{1}{\alpha^2}+3\right)\sigma_{12}^2 = \left(\frac{1+3\alpha^2}{\alpha^2}\right)\sigma_{12}^2$$

So, $$\sigma_e = \frac{\left(1+3\alpha^2\right)^{1/2}}{\alpha}|\sigma_{12}|$$

$$\dot{\varepsilon}_{11}^{PL} = \frac{\sigma_{11}}{\sigma_e}\frac{\dot{\sigma}_e}{h_p} \qquad \dot{\varepsilon}_{12}^{PL} = \frac{3}{2}\frac{\sigma_{12}}{\sigma_e}\frac{\dot{\sigma}_e}{h_p}$$

Now $$\frac{\sigma_{12}}{\sigma_e} = \frac{\alpha}{\sqrt{1+3\alpha^2}}$$

$$\sigma_e = \frac{\sqrt{1+3\alpha^2}}{\alpha}\sigma_{12} = \frac{\sqrt{1+3\alpha^2}}{\alpha}\frac{T}{2\pi R^2 h}$$

So, $$\dot{\varepsilon}_{12}^{PL} = \frac{3}{2}\frac{\cancel{\alpha}}{\cancel{\sqrt{1+3\alpha^2}}}\frac{\cancel{\sqrt{1+3\alpha^2}}}{\cancel{\alpha}}\frac{\dot{T}}{2\pi R^2 h h_p}$$

3b (ii) contd.

$$\Rightarrow \quad \Delta \varepsilon^{PL} = \frac{3}{2} \frac{T - T_Y}{2\pi R^2 h h_p}$$

$T = 2T_Y \quad \Rightarrow$

$$\Delta \varepsilon^{PL}_{12} = \frac{3 T_Y}{4\pi R^2 h} \cdot \left(\frac{E - E_T}{E E_T} \right)$$

$$\text{Now} \quad \dot{\varepsilon}^{PL}_{11} = \frac{S_{11}}{S_{12}} \dot{\varepsilon}^{PL}_{12} = \frac{\frac{2}{3}\sigma_{11}}{\alpha \sigma_{11}} \dot{\varepsilon}^{PL}_{12}$$

$$\Rightarrow \quad \Delta \varepsilon^{PL}_{11} = \frac{2}{3\alpha} \Delta \varepsilon^{PL}_{12}$$

(iii)

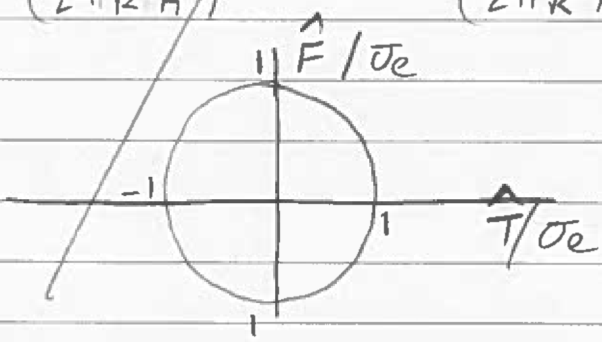
$$\sigma_e^2 = \sigma_{11}^2 + 3\sigma_{12}^2 \Rightarrow \sigma_e = \frac{\sqrt{1 + 3\alpha^2}}{\alpha} |\sigma_{12}|$$

$$\sigma_{12} = \frac{\cancel{2} T_Y}{\cancel{2}\pi R^2 h} \quad \text{at} \quad T = 2T_Y$$

$$\text{So,} \quad \sigma_e = \frac{\sqrt{1 + 3\alpha^2}}{\alpha} \frac{T_Y}{\pi R^2 h}$$

$$\sigma_e^2 = \sigma_{11}^2 + 3\sigma_{12}^2$$

$$\Rightarrow \sigma_e^2 = \left(\frac{F}{2\pi R h} \right)^2 + 3 \left(\frac{T}{2\pi R^2 h} \right)^2 = \hat{F}^2 + \hat{T}^2$$



Examiner: Calculation of plastic strain accumulation was challenging. Strain energy ideas were understood.