Crib for Paper 4C9

Q1. (a) σ_{ij} is stress. $\sigma_{ij,j} = 0$ states force equilibrium with body forces absent (including inertial forces).

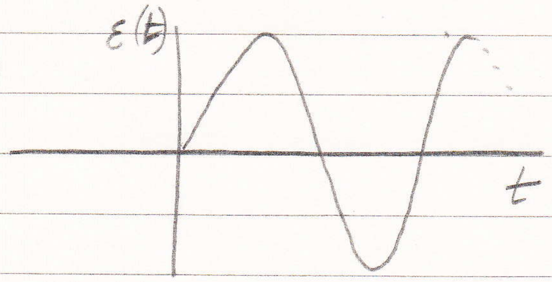
$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}$$

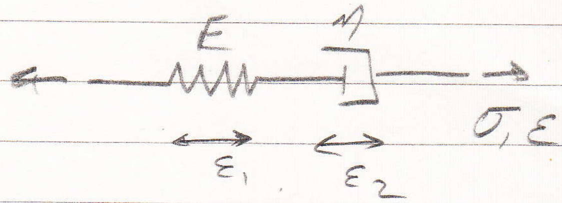
denotes isotropic linear elasticity

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is ~~the~~ small strain strain – displacement relation.

On the surface of body impose tractions $t_i^\circ = \sigma_{ij} n_j$ on S_T and displacements $u_i = u_i^\circ$ on S_u.

(b)





$$\varepsilon = \varepsilon_1 + \varepsilon_2$$

$$\Rightarrow \quad \dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

Now $\sigma = 0$ for $t < 0$

$$\dot{\varepsilon} = A\omega \cos \omega t = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \quad \text{for } t \geq 0$$

Q1(b) contd.

C.F.: $\sigma = B \exp(-t/\tau)$

where $\tau = \eta / E$

$$\dot{\sigma} = -\frac{B}{\tau} \exp(-t/\tau)$$

P.I.: $\sigma = C \cos(\omega t + \phi)$

$$\Rightarrow \dot{\sigma} = -\omega C \sin(\omega t + \phi)$$

$$\frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = -\frac{\omega C}{E} \sin(\omega t + \phi) + \frac{C}{\eta} \cos(\omega t + \phi)$$

$$= \dot{\varepsilon} = \omega A \cos \omega t$$

Alt: $\varepsilon = A \sin \omega t \Rightarrow \varepsilon(t) = \mathrm{Im}\left(A e^{i\omega t}\right)$

$$\sigma = \mathrm{Im}\left(D e^{i\omega t}\right)$$

Write $\sigma = D e^{i\omega t} \Leftrightarrow \dot{\sigma} = i\omega D e^{i\omega t}$

$$\Rightarrow \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = \left(\frac{i\omega D}{E} + \frac{D}{\eta}\right) e^{i\omega t} = A e^{i\omega t}$$

$$\Rightarrow \frac{D}{\eta}(1 + i\omega\tau) = A$$

$$\Rightarrow D = \frac{\eta}{1 + i\omega\tau} . A = \underline{\frac{\eta(1 - i\omega\tau)}{1 + \omega^2\tau^2} . A}$$

$$\Rightarrow |D| = \frac{\eta}{(1 + \omega^2\tau^2)^{1/2}} . A$$

$D = |D| e^{i\phi}$ $\qquad \phi = \tan^{-1}(-\omega\tau)$

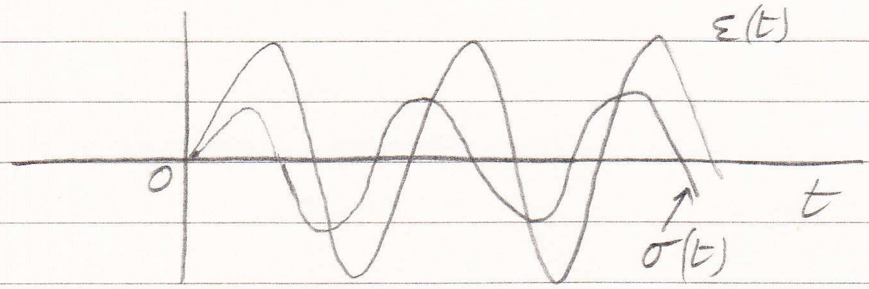
So $\sigma(t)$ lags $\varepsilon(t)$ by ϕ.

Q1. (b) contd.

$$\sigma(t) = B\exp(-t/\tau) + \mathrm{Im}[De^{i\omega t}]$$

$$\sigma(0) = 0 = B + \mathrm{Im}[D] = B - \frac{(\omega\tau)}{1+\omega^2\tau^2}\eta A$$

So $$B = \frac{\omega\tau}{1+\omega^2\tau^2}\eta A$$



Q1. (c) $\sigma_e + c\sigma_h - \sigma_y \leq 0$ and $\dot{\varepsilon}^{pl}_{kk} = 0$

The yield surface is pressure sensitive but plastic flow is incompressible.

Hence normality is not satisfied.

Druckers postulates imply convexity (which is satisfied) and normality (not satisfied).

Comment: First part of question was well done, the visco-elasticity part was more of a challenge. Few understood the implications of the normal to the yield surface.

Q2. (a) (i) $\underline{a} = \nabla \times \nabla \phi$

Now $\nabla \phi = \phi_{,i}\, \underline{e}_i$ $\quad \underline{e}_i = (\underline{e}_1, \underline{e}_2, \underline{e}_3)$

Write $\underline{b} = \nabla \phi = b_i\, \underline{e}_i \iff b_i = \phi_{,i}$

$\underline{a} = \underline{\nabla} \times \underline{b} = \varepsilon_{ijk}\, b_{k,j}\, \underline{e}_i$

$\Rightarrow \quad \underline{a} = \varepsilon_{ijk}\, \phi_{,kj}\, \underline{e}_i \qquad a_i = \varepsilon_{ijk}\, \phi_{,kj}$

But $\varepsilon_{ijk} = -\varepsilon_{ikj}$ and $\phi_{,kj} = \phi_{,jk}$

Hence $a_i = 0 \iff \nabla \times \nabla \phi = 0$.

Q2(a)(ii)

$f = (\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = ?$

$\underline{a} \times \underline{b} = \varepsilon_{ijk}\, a_j\, b_k\, .\underline{e}_i$

$\underline{c} \times \underline{d} = \varepsilon_{ijk}\, c_j\, d_k\, \underline{e}_i = \varepsilon_{ipq}\, c_p\, d_q\, \underline{e}_i$

$\Rightarrow \quad f = \varepsilon_{ijk}\, \varepsilon_{ipq}\, (a_j b_k c_p d_q)$

Now, $\varepsilon_{ijk}\, \varepsilon_{ipq} = \delta_{jp}\,\delta_{kq} - \delta_{jq}\,\delta_{kp}$

$\Rightarrow \quad f = a_j b_k c_j d_k - a_j b_k c_k d_j$

$= (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$

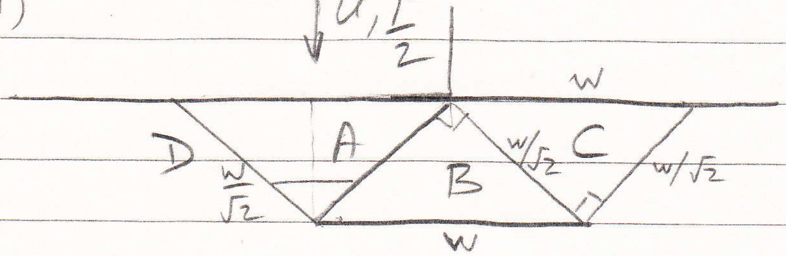
Q2(a)(iii) $\dfrac{\partial \beta}{\partial A_{kl}} = 2\dfrac{\partial A_{ij}}{\partial A_{kl}} A_{ij}$

$A_{ij} = \delta_{ik}\delta_{jl}A_{kl} \Rightarrow \dfrac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}$

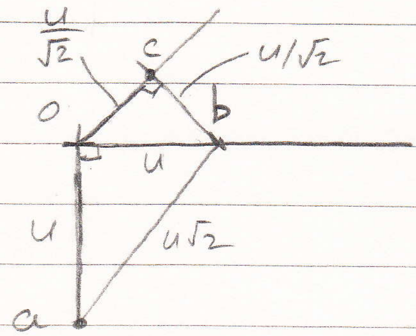
$\Rightarrow \dfrac{\partial \beta}{\partial A_{kl}} = 2A_{kl}$

Q2. (b) (i)

$u = \frac{v}{\sqrt{2}}$

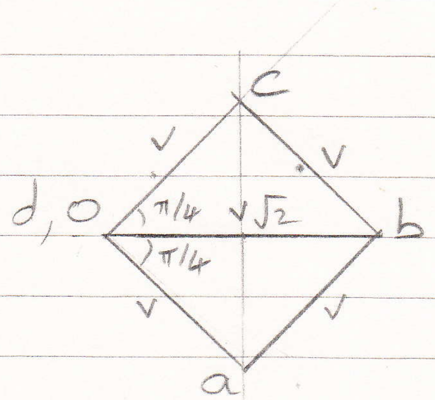


Hodograph :



$$\frac{F}{2}u = kwu\left(1 + 1 + \tfrac{1}{2} + \tfrac{1}{2}\right) \Rightarrow \underline{F = 6kw}$$

(ii)



component along v

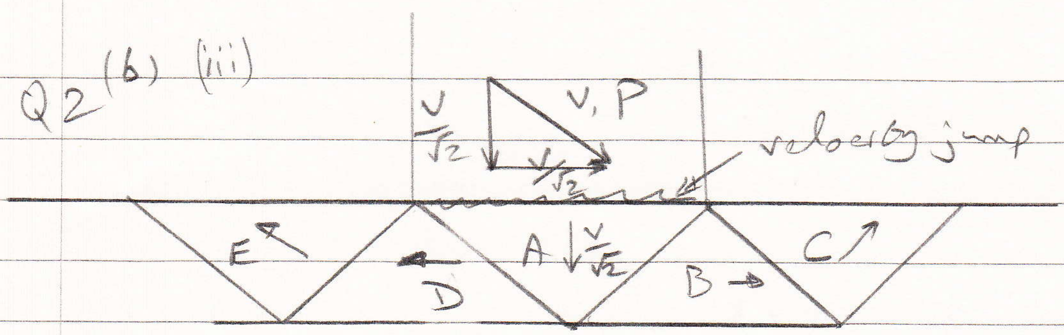
$$Pv = k\left(l_{aD}v_{aD} + l_{AB}v_{AB} + l_{BC}v_{BC} + l_{Co}v_{Co} + l_{Bo}v_{Bo}\right)$$

$$= kwv\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \sqrt{2}\right)$$

$$= \frac{6}{\sqrt{2}}kwv$$

So, $\underline{P = \frac{6}{\sqrt{2}}kw}$

Q2 (b) (iii)



In part (i) we had

$$\dot{W}^{PL} = 6kwu \qquad \text{here } u = \frac{v}{\sqrt{2}}$$

Now $\dot{W}^{PL} = 6kwu + kwu = 7kwu$

So, $Pv = 7kwu = \frac{7}{\sqrt{2}} kwv$

So $P = \frac{7}{\sqrt{2}} kw$

This is greater than part b (ii), so is a worse approximation to the actual solution.

Several candidates struggled with ~~the~~ tensor manipulations. Most could execute the upper bound method.

Q3. (a) Slip line fields satisfy equilibrium in addition to kinematics.
The upper bound method assumes a kinematically admissible field but usually violates equilibrium.
When they coincide, the upper bound satisfies equilibrium.

Neither case may be unique!

(b) (i) $\nabla^4 \Phi = 0$

[Note: few candidates evaluated $\nabla^4 \phi$ to obtain $E = -\frac{2}{3}A$]

$$\sigma_{yy} = \Phi_{,xx} = \frac{A}{3} y^3 + 2Bxy + Cy$$

$$\sigma_{xx} = \Phi_{,yy} = Ax^2 y + Dy + Ey^3$$

$$\sigma_{xy} = -\Phi_{,xy} = -Axy^2 - Bx^2 - Cx$$

$$\nabla^4 \Phi = \Phi_{,xxxx} + 2\Phi_{,xxyy} + \Phi_{,yyyy} = 0$$

$$\Rightarrow \quad 4Ay + 6Ey = 0$$

$$\Rightarrow \quad \underline{E = -\frac{2}{3}A}$$

$\sigma_{yy}(y=h) = q_1 \qquad \sigma_{yy}(y=-h) = q_2$

$\sigma_{xy}(y=\pm h) = 0 \quad \Rightarrow \quad Axh^2 + Bx^2 + Cx = 0$

$$\Rightarrow \quad (Ah^2 + C)x + Bx^2 = 0$$

$$\Rightarrow \quad \underline{B = 0} \quad \text{and} \quad \underline{C = -Ah^2}$$

Q 3 (b) (i) $\sigma_{yy}(y=h) = \frac{A}{3}h^3 + \cancel{2Bxh}^{=0} + Ch = q_1$

$\sigma_{yy}(y=-h) = -\frac{A}{3}h^3 - \cancel{2Bxh}^{=0} - Ch = q_2$

Hence $\underline{q_1 = -q_2}$, and $\underline{A = +\frac{3}{2}\frac{q_2}{h^3}}$

(ii) $\sigma_{xx} = Ax^2y + Dy + Ey^3$

$\Rightarrow \underline{\sigma_{xx} = +\frac{3}{2}\frac{q_2}{h^3}\left(x^2y - \frac{2}{3}y^3\right) + Dy}$

$\sigma_{xy} = Axy^2 + Cx$

$\Rightarrow \underline{\sigma_{xy} = +\frac{3}{2}\frac{q_2}{h^3}\left(xy^2 - h^2x\right)}$

$\sigma_{yy} = \frac{A}{3}y^3 + Cy = \underline{+\frac{3}{2}\frac{q_2}{h^3}\left(\frac{y^3}{3} - h^2y\right)}$

(iii) $M = \int_{-h}^{h} y.\sigma_{xx}\,dy = 0$

$\Rightarrow \int_{-h}^{h}\left[\frac{3}{2}\frac{q_2}{h^3}\left(-\frac{2}{3}y^3\right) + Dy\right]y\,dy = 0$

$\Rightarrow -\frac{2}{5}\frac{q_2}{h^3}h^5 + \frac{2}{3}Dh^3 = 0$

$\Rightarrow \underline{D = \frac{3}{5}\frac{q_2}{h}}$

(iv) Shear force $S = \int_{-h}^{h}\sigma_{xy}\,dy = 0$.

[Most candidates were able to do this.]

4C9 – Assessor's comments

Question 1.
The first part on basics of elasticity theory was poorly attempted by most, with glib answers. The second part on visco-elasticity theory led to few correct answers: this part involved solving a 1st order differential equation. Most candidates knew the significance of Drucker's postulates, but none were able to check for normality in a correct manner.

Question 2.
The first part of the question was about standard tensor manipulations, and it was a surprise that the students found this difficult. The second part was about upper bound calculations and the candidates did this well.

Question 3.
A very popular question on upper bound versus slip line fields, and Airy stress function.
Generally well done, and the main features of the Airy stress function were appreciated.
Some did not use the fact that the stress function must satisfy the biharmonic equation.