new

4C9 2019

$\boxed{\text{Q1}}$ (a)(i) • Correspondence principle : in the Laplace domain, the viscoelastic solution corresponds to the elastic solution, with the substitution $E \to s\bar{E}_r(s)$, $\nu \to s\bar{\nu}_r(s)$ etc (for any time-dependent moduli) [see Data Sheet]

• Applicable ? Applies if there is no time-dependence in the boundary conditions (i.e. boundary between traction, displacement B.C.s). Loading patch radius here is fixed, so it applies.

(ii) For this material, ν is time-independent, so the only time dependence in $\sigma_{11}(t)$, $\sigma_{22}(t)$ and $\sigma_{33}(t)$ is via $p(t)$. So, $p \to p(t)$ gives required time dependence.

(b) If linear elastic:

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

Take Laplace transforms and apply correspondence principal: $E \to s\bar{E}(s)$, $\nu \to \nu$ (time independent)

$$\bar{\varepsilon}_{33}(s) = \frac{1}{s\bar{E}_r(s)}\left[\bar{\sigma}_{33}(s) - \nu\left(\bar{\sigma}_{11}(s) + \bar{\sigma}_{22}(s)\right)\right]$$

Data sheet: $\bar{E}_r(s) = \dfrac{1}{s^2 \bar{J}_c(s)}$

$$\therefore \bar{\varepsilon}_{33}(s) = \bar{J}_c(s)\left[s\bar{\sigma}_{33}(s) - \nu\, s\left(\bar{\sigma}_{11}(s) + \bar{\sigma}_{22}(s)\right)\right]$$

Inverse transform:

$$\varepsilon_{33}(t) = \int_0^t J_c(t-\tau)\,\frac{\partial}{\partial\tau}\left[\sigma_{33}(\tau) - \nu\left(\sigma_{11}(\tau) + \sigma_{22}(\tau)\right)\right] d$$

(c)(i) Substitute for $\sigma_{11}(t)$, $\sigma_{22}(t)$ and $\sigma_{33}(t)$ into the constitutive equation:

$$\varepsilon_{33}(t) = -\frac{1}{\pi a^2}\underbrace{\left(1 - 2\nu\left(\frac{1+2\nu}{2}\right)\right)}_{(1-\nu-2\nu^2)}\int_0^t J_c(t-\tau)\,\frac{\partial p(\tau)}{\partial \tau}\,d\tau$$

all constants

Creep compliance, $J_c(t)$: use Laplace transforms

$$E_r(t) = E e^{-\frac{Et}{\eta}} \Rightarrow \bar{E}_r(s) = \frac{E}{s + \frac{E}{\eta}}$$

Data Sheet

$$\therefore \bar{J}_c(s) = \frac{1}{s^2 \bar{E}_r(s)} = \frac{s + \frac{E}{\eta}}{s^2 E} = \frac{1}{s^2 \eta} + \frac{1}{sE}$$

Inverse transform: $\boxed{J_c(t) = \frac{t}{\eta} + \frac{1}{E}}$

Substitute into constitutive equation:

$$\varepsilon_{33}(t) = -\frac{(1-\nu-2\nu^2)}{\pi a^2}\int_0^t \left(\frac{(t-\tau)}{\eta} + \frac{1}{E}\right)\frac{\partial p(\tau)}{\partial \tau}\,d\tau$$

Loading, for $0 \leq t < t_0$:

- Step response: $p(t) = P_0 H(t)$

$$\therefore \boxed{\varepsilon_{33}(t) = -\frac{(1-\nu-2\nu^2)}{\pi a^2} P_0 \left(\frac{t}{\eta} + \frac{1}{E}\right)}$$

Loading for $t > t_0$:

- Superposition of two step responses: $p(t) = P_0[H(t) - H(t-$

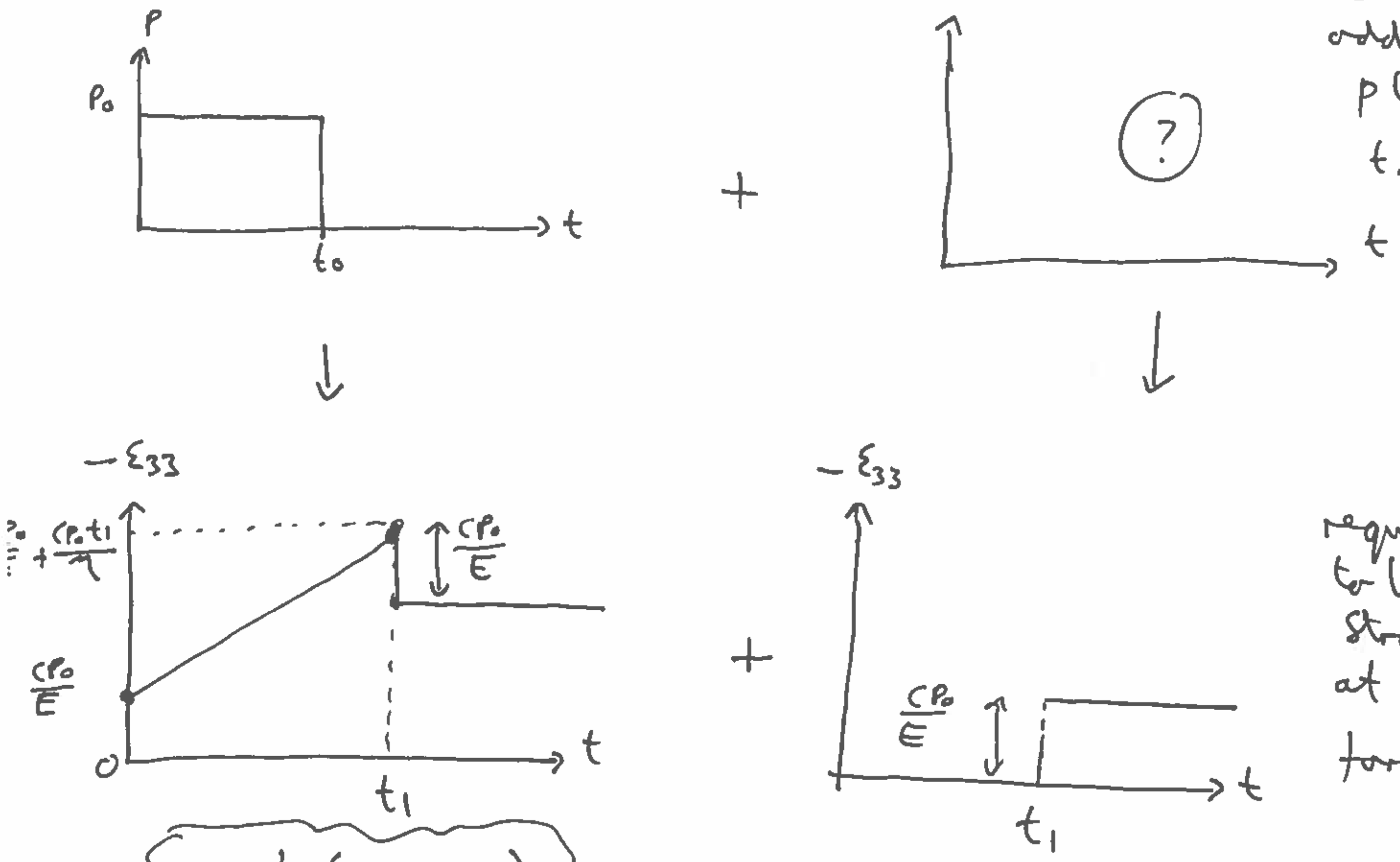
Solution is therefore:

$$\varepsilon_{33}(t) = -\frac{(1-\nu-2\nu^2)}{\pi a^2} P_0\left[\left(\frac{t}{\eta} + \frac{1}{E}\right) - \left(\frac{t-t_1}{\eta} + \frac{1}{E}\right)\right]$$

$$= -\frac{(1-\nu-2\nu^2)}{\pi a^2} P_0\left(\frac{t_1}{\eta}\right)$$

(i.e. constant residual strain:  in dashpot)

(ii) Sketching the response:



unknown additional $p(t)$ for $t \geqslant t_1$

required to hold strain at $\varepsilon_{33}(t_1)$ for $t \geqslant t_1$

$$c = \frac{1}{\pi a^2}(1-\nu-2\nu^2)$$

What $p(t)$ will give $\varepsilon_{33}(t) = -\frac{cP_0}{E} H(t-t_1)$?

Easiest to find using Laplace transform of constitutive relationship :

$$-\frac{cP_0}{E}H(t-t_1) = -c\int_0^t J_c(t-\tau)\frac{\partial p(\tau)}{\partial \tau}\,d\tau$$

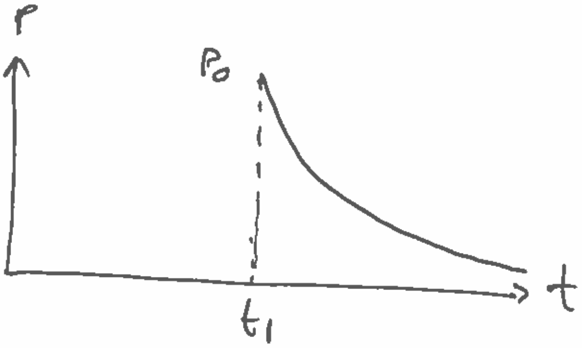
$$\therefore \quad \frac{P_0}{E}\frac{1}{s}e^{-st_1} = \bar{J}_c(s)\; s\,\bar{p}(s)$$

$$= \left(\frac{1}{s^2\eta} + \frac{1}{sE}\right) s\,\bar{p}(s)$$

$$= \frac{s^2\eta + sE}{s^3\eta E}\; s\,\bar{p}(s)$$

$$\therefore \quad \bar{p}(s) = \frac{P_0}{E}\,\frac{\eta E}{s\eta + E}\,e^{-st_1} = P_0\frac{1}{s+\frac{E}{\eta}}$$

Inverse transform : $\boxed{p(t) = P_0 e^{-\frac{E(t-t_1)}{\eta}} H(t-t_1)}$



$\boxed{Q2}$

(a)(i) $\nabla.(\phi \underline{u}) = \frac{\partial}{\partial x_i}(\phi u_i) = \frac{\partial \phi}{\partial x_i} u_i + \phi \frac{\partial u_i}{\partial x_i}$

$= (\nabla \phi).\underline{u} + \phi \nabla.\underline{u}$

(ii) $\nabla.\nabla\times\underline{u} = \frac{\partial}{\partial x_i}[\nabla\times\underline{u}]_i = \frac{\partial}{\partial x_i}\left[\underset{\nwarrow \text{a constant}}{e_{ijk}} \frac{\partial}{\partial x_j} u_k\right]$

$= e_{ijk} \frac{\partial^2 u_k}{\partial x_i \partial x_j}$

Given that : $\frac{\partial^2 u_k}{\partial x_i \partial x_j} = \frac{\partial^2 u_k}{\partial x_j \partial x_i}$ for any i, j

And : $e_{ijk} = -e_{ikj}$ for any i, j

$\Rightarrow e_{ijk} \frac{\partial^2 u_k}{\partial x_i \partial x_j} = 0$

(iii) $[\nabla\times(\nabla\times\underline{u})]_i = e_{ijk} \frac{\partial}{\partial x_j}[\nabla\times\underline{u}]_k$

$= e_{ijk} \frac{\partial}{\partial x_j}\left[e_{kpq} \frac{\partial}{\partial x_p} u_q\right]$

$= e_{kij} e_{kpq} \frac{\partial^2 u_q}{\partial x_j \partial x_p}$

Data Sheet $\longrightarrow$ $= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \frac{\partial^2 u_q}{\partial x_j \partial x_p}$

$= \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j}$

$= [\nabla(\nabla.\underline{u}) - (\nabla.\nabla)\underline{u}]_i$

(b)(i) Minimum PE (Data Sheet):

$$\delta\Pi = \int_V \delta U \, dV - \int_S t_i^e \, \delta u_i \, dS - \int_V \overset{\text{neglect}}{\cancel{b_i \delta u_i}} \, dV = 0$$

Strain energy density:

- non-zero strain components : ε_{11} only

$$\therefore \; U = \frac{E}{2(1+\nu)}\left[\varepsilon_{11}^2 + \frac{\nu}{1-\nu}\varepsilon_{11}^2\right] = \frac{E}{2(1+\nu)(1-\nu)}\varepsilon_{11}^2$$

- displacements required, so write $\varepsilon_{11} = u_{1,1}$
- note $E = E(x_1)$

$$\therefore \; U = \frac{E(x_1)}{2(1+\nu)(1-\nu)}\,(u_{1,1})^2$$

- variation in U:

$$\delta U = \frac{\partial U}{\partial u_{1,1}}\,\delta u_{1,1} = \frac{E(x_1)}{(1+\nu)(1-\nu)}\,u_{1,1}\,\delta u_{1,1}$$

- per unit area of elastic layer:

$$\int_V \delta U \, dV = \int_0^H \frac{E(x_1)}{(1+\nu)(1-\nu)}\,u_{1,1}\,\delta u_{1,1}\,dx_1$$

- integration by parts:

$$\int_V \delta U \, dV = \left[\frac{E(x_1)}{(1+\nu)(1-\nu)}\,u_{1,1}\,\delta u_1\right]_0^H \cancel{-\int_0^H \frac{1}{(1+\nu)(1-\nu)}}$$

$$-\int_0^H \frac{1}{(1+\nu)(1-\nu)}\,\underbrace{\frac{d}{dx_1}\Big[E(x_1)\,u_{1,1}\Big]}_{=\left(\frac{dE}{dx_1}u_{1,1} + E u_{1,11}\right)}\,\delta u_1\,dx_1$$

$\delta u_1(H) = \delta w = 0$ (w fixed)

$\delta u_1(0) = 0$

③

External tractions and displacements:

- Let $\underline{F} = F\underline{e}_1$ be the force per unit area acting on the top block, to cause displacement $w\underline{e}_1$
- Per unit area:

$$\int_S t_i^e \delta u_i \, dS = F \delta w = 0 \quad (w \text{ fixed})$$

Minimum PE

If $\delta\pi = 0$:

$$\cancel{\left[\frac{E(H)}{\cancel{(1+\nu)(1-\nu)}} u_{1,1}(H) - F \right]}^{\,0} \boxed{\delta w} = 0$$

$$- \frac{1}{\cancel{(1+\nu)(1-\nu)}} \int_0^H \left(\frac{dE}{dx_1} u_{1,1} + E u_{1,11} \right) \delta u_1 \, dx_1 = 0$$

∴ Governing equations:

$$\boxed{\frac{dE}{dx_1} u_{1,1} + E u_{1,11} = 0} \quad ① \qquad 0 \leq x_1 \leq H$$

Boundary conditions: $\cancel{\boxed{F = E(H) u_{1,1}(H)}} \; ②$ $\qquad x_1 = H$

~~$u_1(H) = w$ ③~~
~~$u_1(0) = 0$ ④ given~~

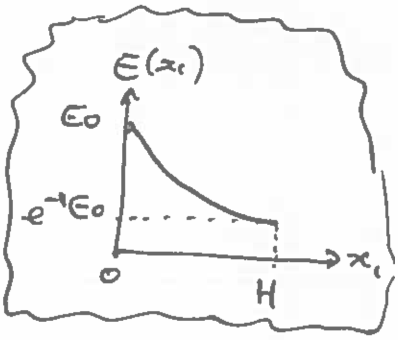
$$\left. \begin{array}{l} u_1(H) = w \\ u_1(0) = 0 \end{array} \right\} \text{given}$$

From constitutive equations, force per unit area on top plate, to displace w:

$$\sigma_{11} = F = \frac{\partial U}{\partial \varepsilon_{11}} = \frac{E}{(1+\nu)(1-\nu)} \varepsilon_{11} = \frac{E}{(1+\nu)(1-\nu)} u_{1,1}(H) \qquad ②$$

(ii) For

$$\left.\begin{aligned} E(x_1) &= E_0 e^{-\frac{x_1}{H}} \\ \frac{dE}{dx_1} &= -\frac{E_0}{H} e^{-\frac{x_1}{H}} \end{aligned}\right\} \text{sub. into ①}$$



$$\therefore \boxed{-\frac{1}{H}u_{1,1} + u_{1,11} = 0}$$

Try: $u_1 = a + b e^{-cx_1}$

$$\Rightarrow \frac{c}{H}e^{-cx_1} + c^2 e^{-cx_1} = 0 \qquad \therefore c = -\frac{1}{H}$$

To satisfy B.C ③: $a + b e^{-1} = w$

B.c. ④ $a + b = 0$

$$\therefore a = \left(\frac{w}{1-e^{-1}}\right) = -b$$

$$\therefore \boxed{u_1 = \left(\frac{w}{1-e^{-1}}\right)\left(1 - e^{-\frac{x_1}{H}}\right)}$$

Using ②: $F = \left(\frac{E_0 e^{-1}}{(1+\nu)(1-\nu)}\right)\left(\frac{w}{1-e^{-1}}\right)\frac{1}{H}e^{-1}$

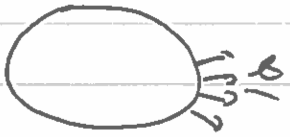
$$\therefore \boxed{F = \frac{E_0}{(1+\nu)(1-\nu)}\frac{w}{H}\left(\frac{e^{-2}}{1-e^{-1}}\right) \approx 0.21\left(\frac{w}{H}\right)\frac{E_0}{(1+\nu)(1-\nu)}}$$

3.

a) $$e_{ijk}\sigma_{jk} = \begin{bmatrix} e_{1jk}\sigma_{jk} \\ e_{2jk}\sigma_{jk} \\ e_{3jk}\sigma_{jk} \end{bmatrix} - \begin{bmatrix} \sigma_{23} - \sigma_{32} \\ \sigma_{31} - \sigma_{13} \\ \sigma_{12} - \sigma_{21} \end{bmatrix}$$

$$= \underline{0} \text{ if } \sigma_{ij} = \sigma_{ji}$$

b) Linear balance



$\underline{t} = \underline{\sigma}\,\underline{n}$

$$\int_{\partial\Omega} \underline{\sigma} n \, ds + \int_{\Omega} \underline{f} \, dx = 0$$

$$\int_{\Omega} \nabla \cdot \underline{\sigma} \, dx + \int_{\Omega} f \, dx = 0$$

$\Rightarrow \quad -\nabla \cdot \underline{\sigma} = \underline{f}$ by localisation argument.

Angular balance (moment)

$$\int_{\partial\Omega} \underbrace{\underline{x} \times \underline{\sigma} n}_{(e_{ijk} x_j \sigma_{kl}) n_l} \, ds + \int_{\Omega} \underline{x} \times \underline{f} \, dx = 0$$

$$\int_{\Omega} \frac{\partial}{\partial x_l} (e_{ijk} x_j \sigma_{kl}) \, dx + \int_{\Omega} \underline{x} \times \underline{f} \, dx = 0$$

$$= \int_{\Omega} e_{ijk} x_j \frac{\partial \sigma_{kl}}{\partial x_l} + e_{ijk} \delta_{jl} \sigma_{kl} \, dx + \int_{\Omega} \underline{x} \times \underline{f} \, dx = 0$$

$$\int_{\Omega} \underline{x} \times \underbrace{(\nabla \cdot \underline{\sigma} + f)}_{= 0 \text{ by linear balance}} \, dx + \int_{\Omega} \underbrace{e_{ijk} \sigma_{kj}}_{\underline{\underline{\varepsilon}} : \underline{\underline{\sigma}}^T = 0 \text{ if } \underline{\sigma} = \underline{\sigma}^T} \, dx$$

c) $\underline{\underline{l}} = \dfrac{\partial \underline{v}}{\partial \underline{x}} = \dfrac{\partial \underline{v}}{\partial \underline{X}} \cdot \dfrac{\partial \underline{X}}{\partial \underline{x}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$

d) i) $\displaystyle\int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{d}}\; dx = \int_{\Omega_0} J\underline{\underline{\sigma}} : \underline{\underline{d}}\; dX$

$\therefore \quad \underline{\underline{\tau}} = J\underline{\underline{\sigma}} \quad$ when $\quad J = \det \underline{\underline{F}}$

ii) $\displaystyle\int \underline{\underline{\sigma}} : \underline{\underline{d}}\; dx = \int_{\Omega_0} J\underline{\underline{\sigma}} : \left(\dot{F}F^{-1} + F^{-T}\dot{F}^{T}\right) dX$

$\displaystyle = \int_{\Omega_0} J\underline{\underline{\sigma}} : \dot{F}F^{-1}\, dX$ (by symmetry of $\underline{\underline{\sigma}}$)

$\displaystyle = \int_{\Omega_0} J\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T} : \dot{\underline{\underline{F}}}\; dX$

$\displaystyle = \int J\sigma\underline{\underline{F}}^{-T} : \underline{\underline{F}}^{-T}\underline{\underline{F}}^{T}\dot{\underline{\underline{F}}}\; dX$

$\displaystyle = \int \underbrace{J\underline{\underline{F}}^{-1}\underline{\underline{\sigma}}\,\underline{\underline{F}}^{-T}}_{=\,\underline{\underline{S}}} : F^{T}\dot{F}\; dX$

~~since $\dot{\underline{\underline{E}}} = \underline{\underline{F}}^{T}\dot{\underline{\underline{F}}} + \dot{\underline{\underline{F}}}^{T}\underline{\underline{E}}$~~

since $\dot{\underline{\underline{E}}} = \underline{\underline{F}}^{T}\dot{\underline{\underline{F}}} + \dot{\underline{\underline{F}}}^{T}\underline{\underline{F}}$ is symmetric.

c) i) $\tilde{\underline{\underline{V}}} = \underline{\underline{Q}}\,\underline{\underline{V}}\,\underline{\underline{Q}}^T$

$$\dot{\tilde{\underline{\underline{V}}}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{V}}\,\underline{\underline{Q}}^T + \underline{\underline{Q}}\,\dot{\underline{\underline{V}}}\,\underline{\underline{Q}}^T + \underline{\underline{Q}}\,\underline{\underline{V}}\,\dot{\underline{\underline{Q}}}^T \neq \underline{\underline{Q}}\,\dot{\underline{\underline{V}}}\,\underline{\underline{Q}}^T$$

$\therefore$ $\dot{\underline{\underline{V}}}$ is not objective.

ii) $\tilde{\underline{\underline{l}}} = \dot{\tilde{\underline{\underline{F}}}}\,\tilde{\underline{\underline{F}}}^{-1}$, $\quad \underline{\underline{l}} = \dot{\underline{\underline{F}}}\,\underline{\underline{F}}^{-1}$, $\quad \tilde{\underline{\underline{F}}} = \underline{\underline{Q}}\,\underline{\underline{F}}$

$$\dot{\tilde{\underline{\underline{F}}}} = \dot{\underline{\underline{Q}}}\,\underline{\underline{F}} + \underline{\underline{Q}}\,\dot{\underline{\underline{F}}}$$

$$\hat{l} = (\dot{\underline{\underline{Q}}}\,\underline{\underline{F}} + \underline{\underline{Q}}\,\dot{\underline{\underline{F}}})(\underline{\underline{Q}}\,\underline{\underline{F}})^{-1}$$

$$= (\dot{\underline{\underline{Q}}}\,\underline{\underline{F}} + \underline{\underline{Q}}\,\dot{\underline{\underline{F}}})(\underline{\underline{F}}^{-1}\underline{\underline{Q}}^{-1})$$

$$= \dot{\underline{\underline{Q}}}\,\underline{\underline{F}}\,\underline{\underline{F}}^{-1}\underline{\underline{Q}}^T + \underline{\underline{Q}}\,\dot{F}F^{-1}\underline{\underline{Q}}^4$$

$$= \dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T + \underline{\underline{Q}}\,\underline{\underline{l}}\,\underline{\underline{Q}}^T \neq \underline{\underline{Q}}\,\underline{\underline{l}}\,\underline{\underline{Q}}^T$$

$\therefore$ not objective.

$$\tilde{\underline{\underline{d}}} = (\tilde{\underline{\underline{l}}} + \tilde{\underline{\underline{l}}}^T)/2 = (\dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T + \underline{\underline{Q}}\,\dot{\underline{\underline{Q}}}^T)/2 + \underline{\underline{Q}}(\underline{\underline{l}} + \underline{\underline{l}}^T)\underline{\underline{Q}}^T/2$$

$$= (\dot{\underline{\underline{Q}}}\,\underline{\underline{Q}}^T + \underline{\underline{Q}}\,\dot{\underline{\underline{Q}}}^T)/2 + \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T$$

Consider $\underline{\underline{Q}}\,\underline{\underline{Q}}^T = \underline{\underline{I}}$

$$\frac{d}{dt}(\underline{\underline{Q}}\,\underline{\underline{Q}}^T) = \dot{\underline{\underline{Q}}}\,Q^T + \underline{\underline{Q}}\,\dot{\underline{\underline{Q}}}^T = 0$$

$\therefore$ $\tilde{d} = \underline{\underline{Q}}\,\underline{\underline{d}}\,\underline{\underline{Q}}^T \Rightarrow$ objective.

MODULE 4C9: Examiner's comments

Question 1 (viscoelasticity)

Parts (a) and (b) were generally well-answered. Few candidates answered (c)(i) correctly, with problems with the inverse Laplace transform and failure to spot the superposition of two step responses common. Fewer answered (c)(ii) correctly and many did not attempt this part. A surprising number of answers for (c) did not involve time.

Question 2 (variational methods)

The tensor manipulations in (a) were generally performed without difficulty. There was general discomfort with process behind variational methods, with few able to derive the governing differential equation. A number of solutions failed to account for the contribution of the spatially varying Young's modulus.

Question 3 (nonlinear continuum mechanics)

This question was very well done by almost all who attempted it. The number and quality of the attempts was pleasing given that this is the first year that the topic has been included in the course.