# 4F1 crib 2019

1 (a) Bookwork.

(b)(i) $$m\ddot{y} = k(x-y) \Rightarrow (ms^2+k)\bar{y} = k\bar{x}$$

$$\Rightarrow \frac{\bar{y}}{\bar{x}} = \frac{k}{ms^2+k}$$

Hence

$$T_{\bar{x}\to\bar{f}} = \frac{k(\bar{x}-\bar{y})}{\bar{x}} = k\left(1-\frac{k}{ms^2+k}\right) = \frac{kms^2}{ms^2+k}$$

(ii) Closed loop poles are roots of:

$$ms^2+k+\alpha kms^2 = (1+\alpha k)ms^2+k = 0$$

for a proportional gain of $\alpha$. Roots are always on $j\omega$-axis, or one is in RHP (if $1+\alpha k < 0$).

(iii) Closed loop poles are roots of:

$$(ms^2+k)(\tau s+1) + \alpha kms^2$$

$$= m\tau s^3 + (m+\alpha km)s^2 + k\tau s + k$$

$$= a_0 s^3 + a_1 s^2 + a_2 s + a_3$$

Since $m, k, \tau > 0$, from Routh-Hurwitz criterion, all roots are in the LHP if and only if

$$a_1 a_2 - a_0 a_3 = (m+\alpha km)k\tau - m\tau k$$

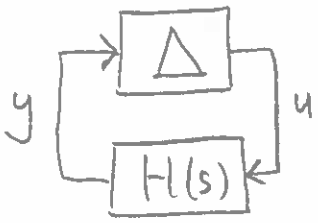
$$= \alpha k^2 m\tau > 0 \quad \Leftrightarrow \quad \alpha > 0$$

(iv) $f = k(x-y)$ ✓

$$\bar{y} = \frac{1}{1+\Delta}\frac{1}{m_0 s^2}\bar{f} \quad \Rightarrow \quad m\ddot{y} = f \quad ✓$$

These are the same equations as used in (b)(i), hence the result.

(v) Set $r=0$ in Fig. 2 and re-draw as:




To find $H(s)$ set $\Delta = 0$

$$\left.\begin{aligned} \bar{y} &= -\bar{u} + \frac{1}{m_0 s^2}\bar{f} \\ \bar{f} &= k(\bar{x}-\bar{y}) \\ \bar{x} &= -G_a K \bar{f} \end{aligned}\right\} \quad \begin{aligned} &\Rightarrow \bar{f} = k(-G_a K \bar{f} - \bar{y}) \\ &\Rightarrow \bar{f} = \frac{-k\bar{y}}{1+kG_aK} \\ &\Rightarrow \bar{y}\left(1 + \frac{1}{m_0 s^2}\frac{k}{1+kG_aK}\right) = -\bar{u} \end{aligned}$$

Hence

$$H(s) = \frac{\bar{y}}{\bar{u}} = \frac{-m_0 s^2(1+kG_aK)}{m_0 s^2(1+kG_aK)+k}$$

By the Small Gain Theorem Fig. 2 is stable for all $\Delta(s)$ satisfying $|\Delta(j\omega)| \leq h$ if and only if $|H(j\omega)| < h^{-1}$ for all $\omega$.

(vi) $H(\infty) = -1$ when $K(s) = \alpha$, hence $h < 1$.

$m = m_0(1+\Delta)$ clearly can't be negative but there is an upper bound on predicted $m$ to stabilise Fig. 2. This is more conservative than Part (b)(iii), as expected since Small Gain approach assumes a frequency dependent $\Delta$.

**Q1. Robustness and small gain theorem for micro-mechanical system.**
15 attempts, mean 13.9/20 (69.7%), maximum 19, minimum 8.
Candidates were generally able to give correct solutions to Part (a) and Parts (b)(i)–(b)(iv). Parts (v) and (vi) caused more difficulties, especially the comparison of the small gain theorem robustness test to the result of Part (b)(iii) which few candidates answered sensibly.

Version MCS

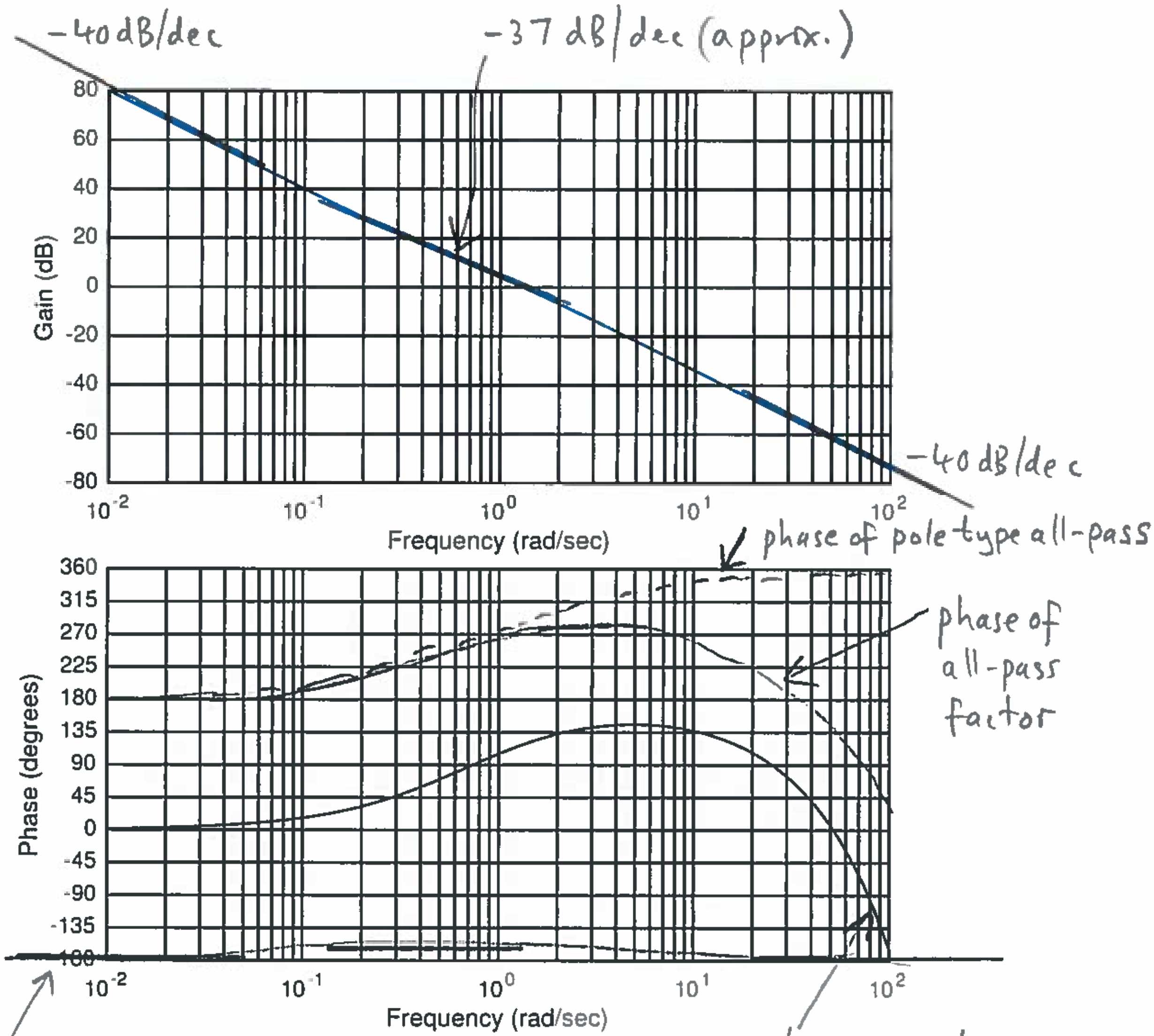
Candidate Number:

EGT3

ENGINEERING TRIPOS PART IIB

?, Module 4F1, Question ??.

2 (a)(i)




Extra copy of Bode diagram

min. phase close to −180°

shape suggests a time delay

(a)(ii) all-pass factor $= -\dfrac{p+s}{p-s}\, e^{-sT}$

sketch of probable factor without time delay suggests $p \simeq 1$

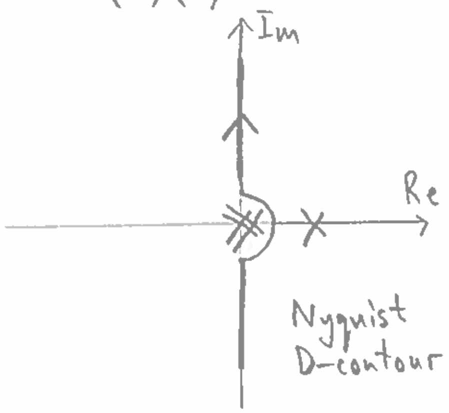
phase lag of about 330° from time delay at 100 rad/sec

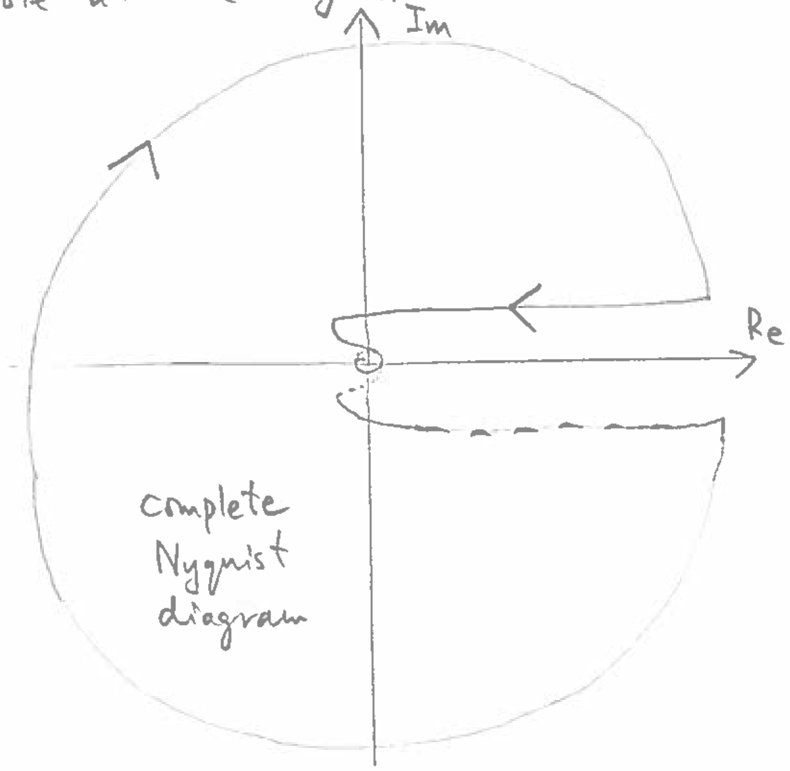
$\Rightarrow \quad 100T = 330 \times 2\pi/360 \quad \Rightarrow \quad T \simeq 5.8 \times 10^{-2}$

2 (a) (iii) For conventional loop shape cross-over should not be much lower than 1 rad/sec because of RHP pole, and should not be much greater than about $\pi/(2T) \simeq 27$ rad/sec, where phase lag from time delay is $90°$.

(iv) Slope at low freqs. = −40 dB/dec which suggests a double pole at the origin.

(b) (i)




(ii)
Need 1 anti-clockwise encirclement about $-1/k$ for closed-loop stability. Not satisfied: 0 encirclements for $k < 0$ and 1 clockwise (or more) when $k > 0$.

(c) (i) Peak phase is around 5 rad/sec. Approach: increase phase by close to $90°$ at around that frequency and make that the gain cross-over frequency. Choose $K(s) = k\,K_1(s)^2$ where

$$K_1(s) = \frac{\alpha\,(s + w_c/\alpha)}{s + w_c\alpha}$$

$\alpha = \sqrt{5}$, $w_c = 5$ gives peak phase advance of $41.8°$ at 5 rad/sec

$\Rightarrow 83.6°$ phase lead - sufficient to take phase above $225°$ .. ..

Need $20\log_{10} k = 22$ approx to make 5 rad/sec the gain crossover

$\Rightarrow k = 12.6$

2(c)(i) cont.

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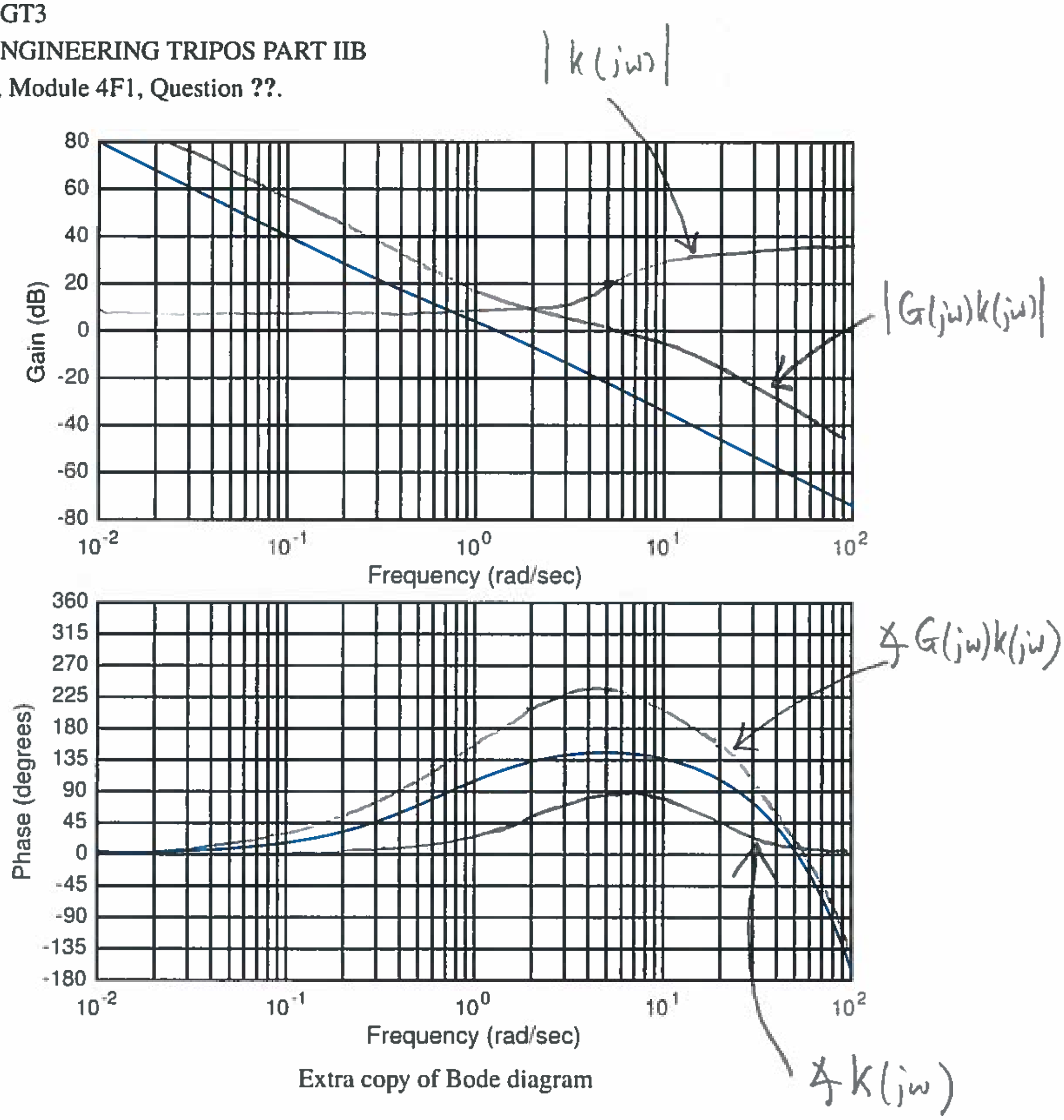
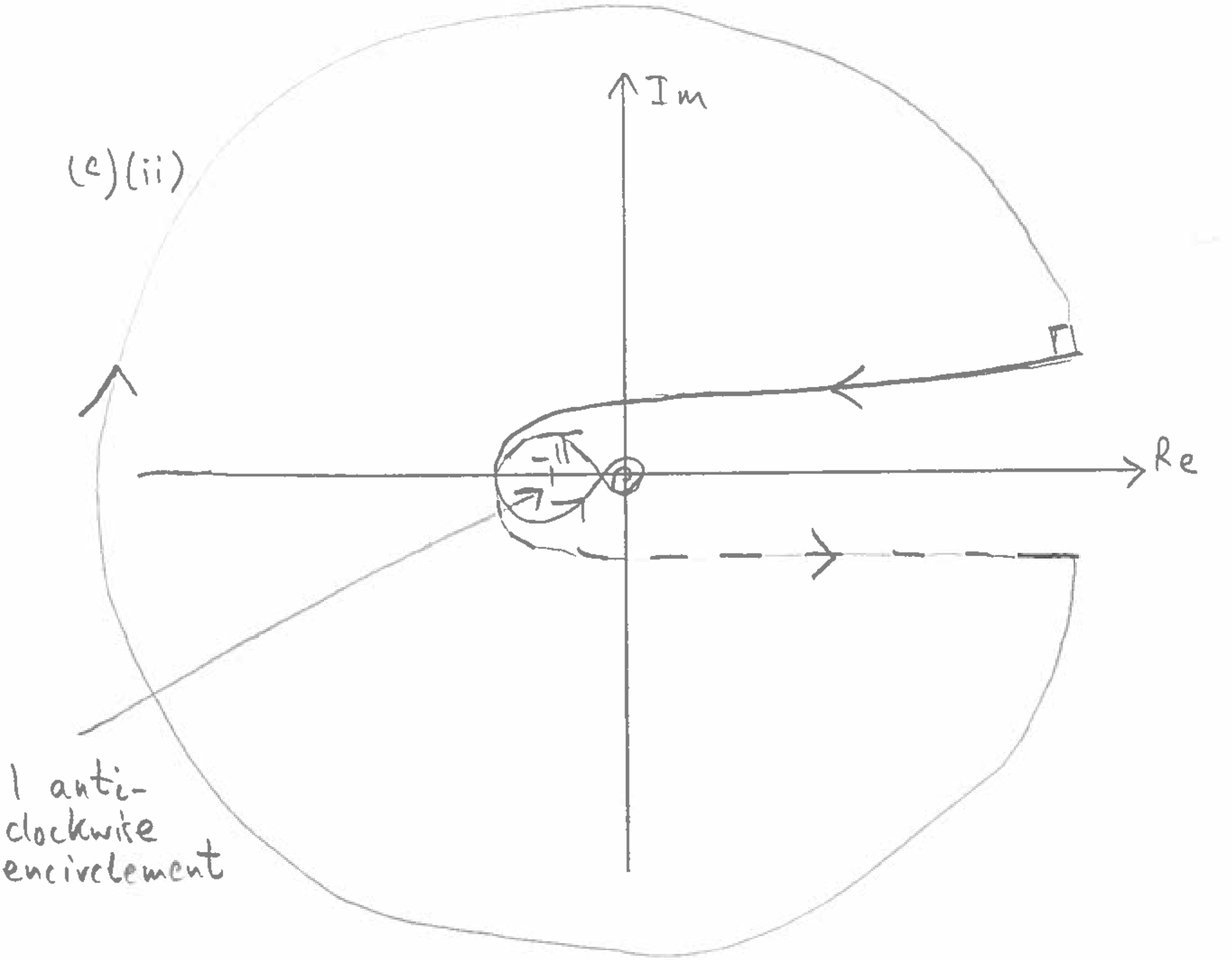



Extra copy of Bode diagram

(c)(ii)




Nyquist diagram of $G(s)k(s)$ (not to scale)

Note there is one anti-clockwise encirclement of $-1$ point as required for closed-loop stability

(iii) Phase margin just about $45°$.

Extra time delay: $\omega T_0 = \frac{\pi}{4} \Rightarrow T_0 = \frac{\pi}{20} = 0.157$ sec

[Actual transfer function of $G(s)$ was $\frac{2s+1}{s^2(s-1)} e^{-0.06s}$]

2(a)(i)

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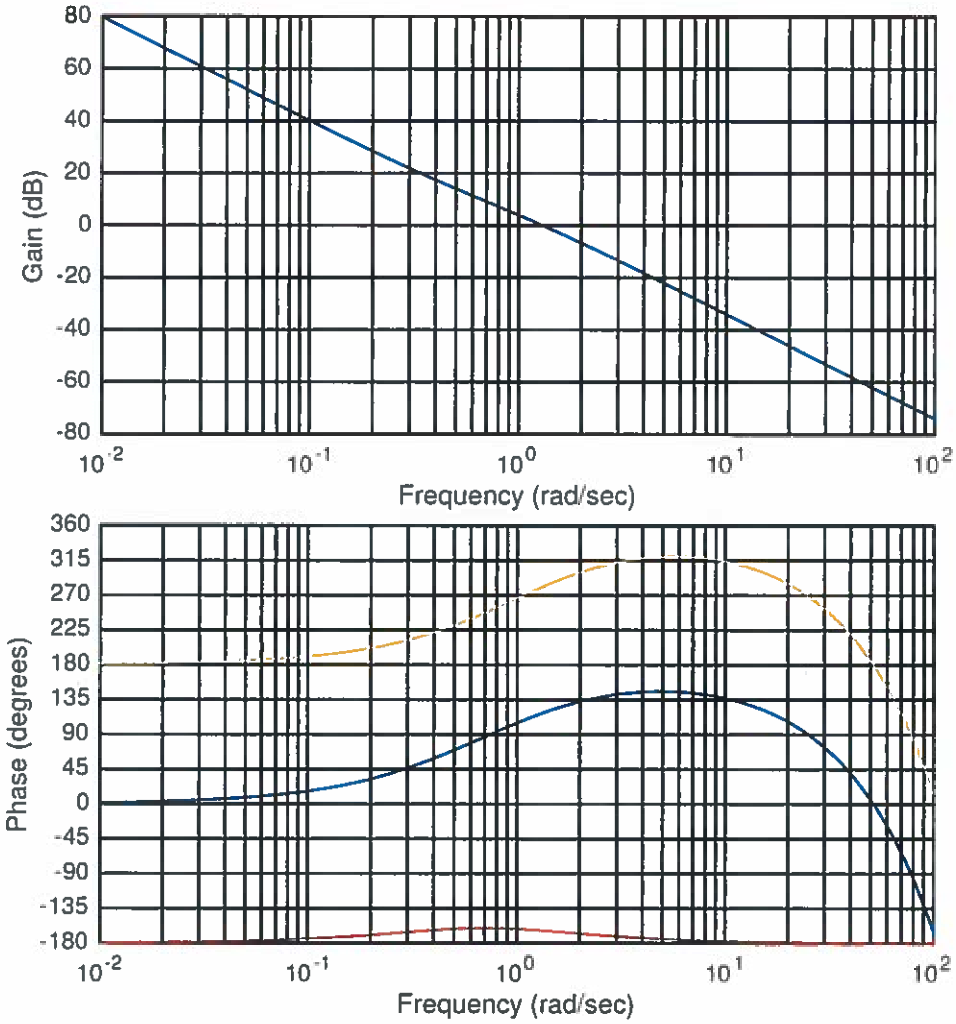
?, Module 4F1, Question ??.




Extra copy of Bode diagram

2(c)(i)

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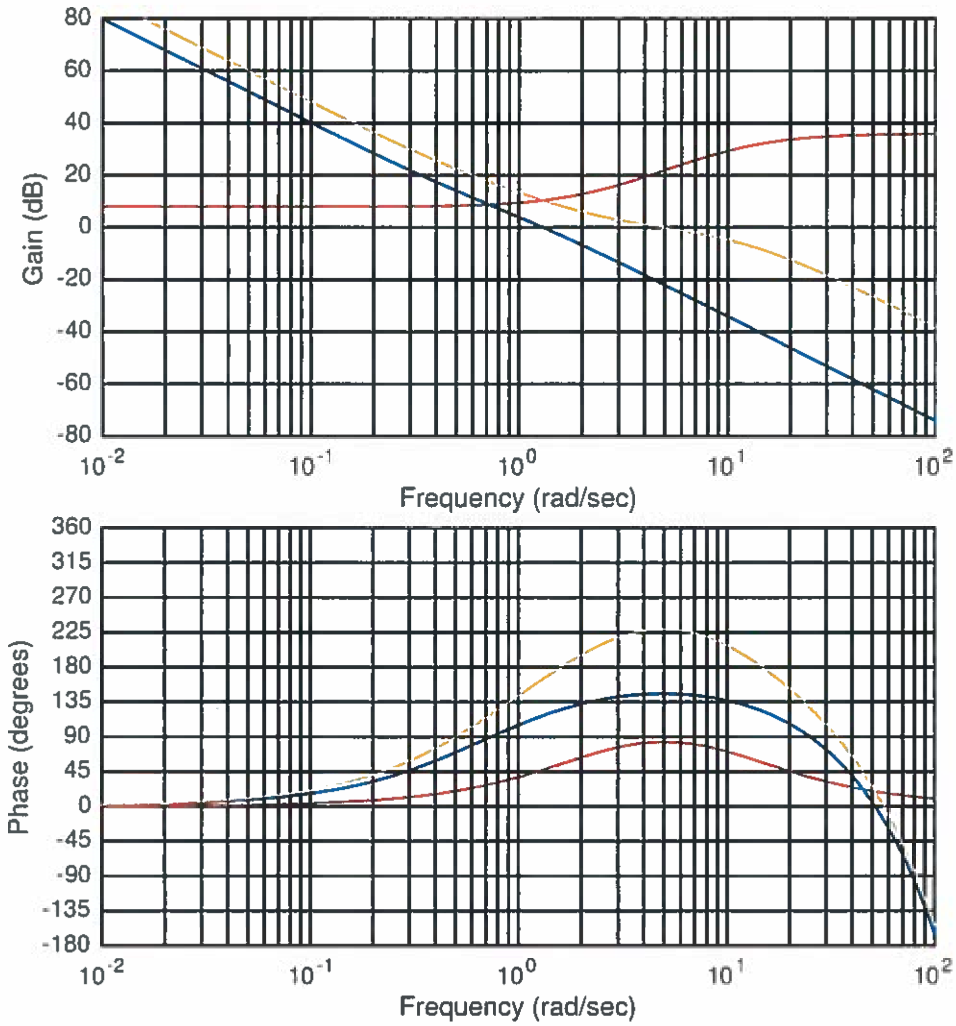
ENGINEERING TRIPOS PART IIB
?, Module 4F1, Question ??.




Extra copy of Bode diagram

**Q2. Bode gain-phase relations and compensator design.**
10 attempts, mean 14.8/20 (74%), maximum 18, minimum 11.
The least popular question but with mostly very good solutions. Not many candidates spotted that the stable all-pass factor in (a)(i) was a time delay, and there were a lot of errors in computing the smallest time delay in (c)(iii).

3(a)(i) $\bar{y}(s) = G(s)\frac{1}{s}$ $G(0) = 1$

$$\dot{y}(0^+) = \lim_{s\to\infty} s\,\mathcal{L}(\dot{y}(t))$$ INITIAL VALUE THEOREM

$$= \lim_{s\to\infty} s^2\bar{y}(s) = \lim_{s\to\infty} sG(s)$$

$$= \frac{a_1 a_2 \dots a_{n-1}}{b_n} \qquad (b_n > 0)$$

initial undershoot $\Leftrightarrow$ $\dot{y}(0^+) < 0$

$\Leftrightarrow$ odd number of $a_i$ are negative

(ii) $\bar{y}(s) = G(s)\frac{1}{s} \Rightarrow \bar{y}(-1/a_i) = 0$

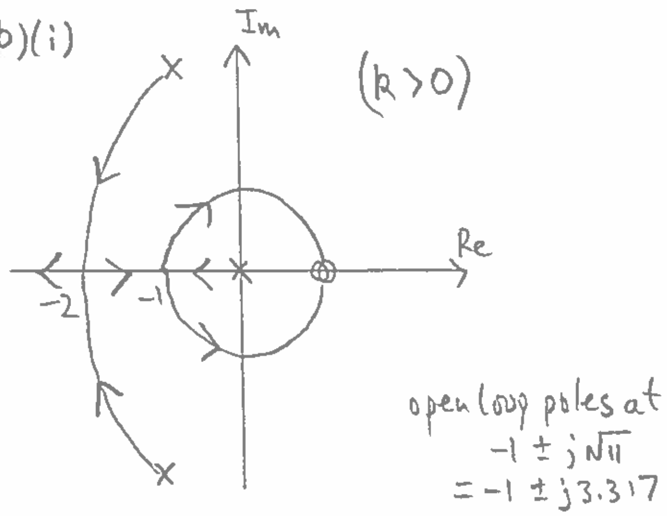
$$\bar{y}(s) = \int_0^\infty y(t)e^{-st}\,dt$$

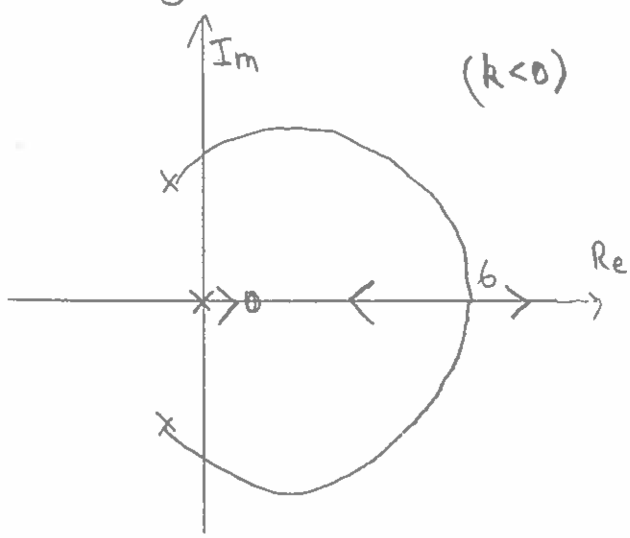
Since $a_i < 0$, $-1/a_i$ is in the region of convergence of the transform. Hence

$$0 = \int_0^\infty y(t)\,e^{t/a_i}\,dt$$

(iii) The integral in (ii) implies that $y(t)$ will be positive at some times and negative at others. Hence there must be at least one $t_1 > 0$ where $y(t_1) = 0$.

(b)(i)



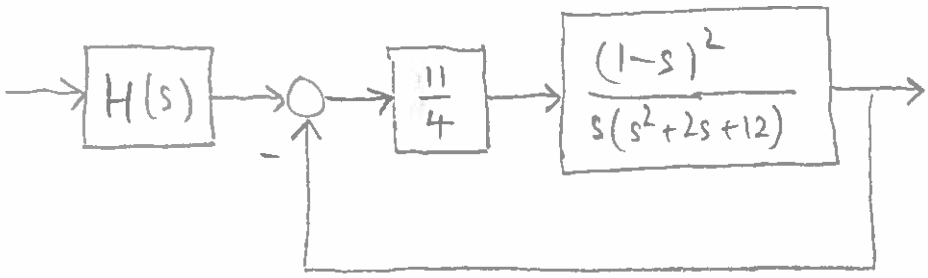
(b)(ii) $\quad G(-1) = \dfrac{4}{-1(11)} \quad \Rightarrow \quad k = \dfrac{11}{4}$

Since $-1$ is a breakaway point this should be a double root. Hence

$$s(s^2+2s+12) + \frac{11}{4}(s-1)^2 \equiv (s+1)^2(s+c)$$

$$\Rightarrow c = \frac{11}{4} \text{, so poles are at: } -1, -1, -\frac{11}{4}$$

(iii)



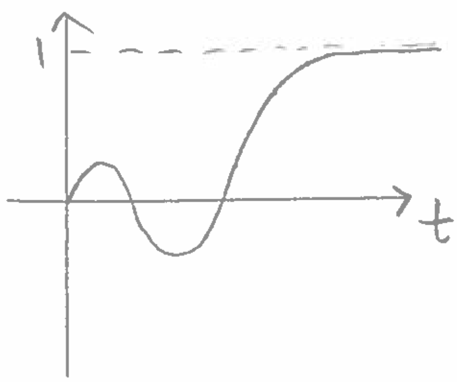

Need: $\dfrac{(1-s)^2}{(s+1)^3} = H(s)\,\dfrac{\frac{11}{4}(1-s)^2}{s(s^2+2s+12)+\frac{11}{4}(1-s)^2}$

$$= H(s)\,\frac{\frac{11}{4}(1-s)^2}{(s+1)^2(s+\frac{11}{4})}$$

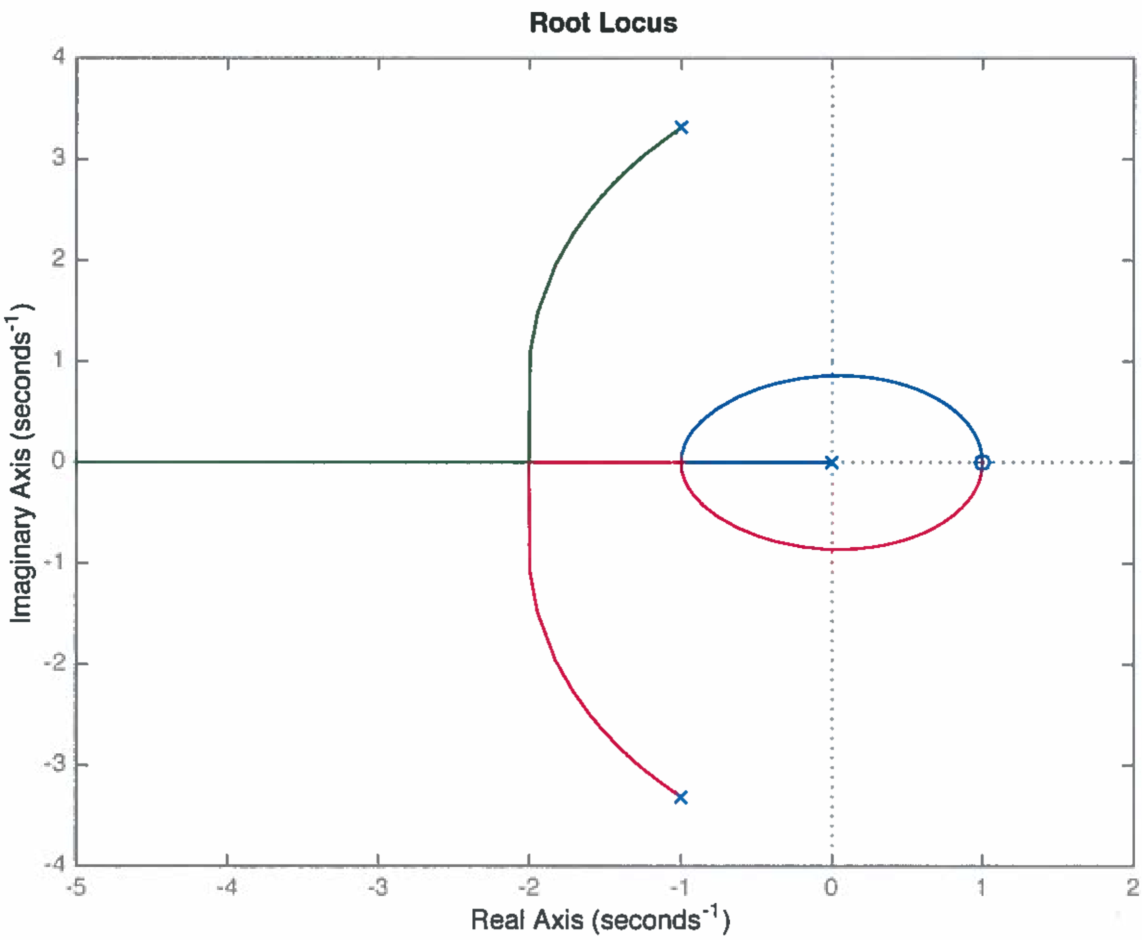
$$\Rightarrow H(s) = \frac{\frac{4}{11}(s+\frac{11}{4})}{s+1} = \frac{\frac{4s}{11}+1}{s+1}$$

(iv) From (a)(i) initial slope $>0$. From (a)(iii) at least one zero crossing. DC gain $= 1$.

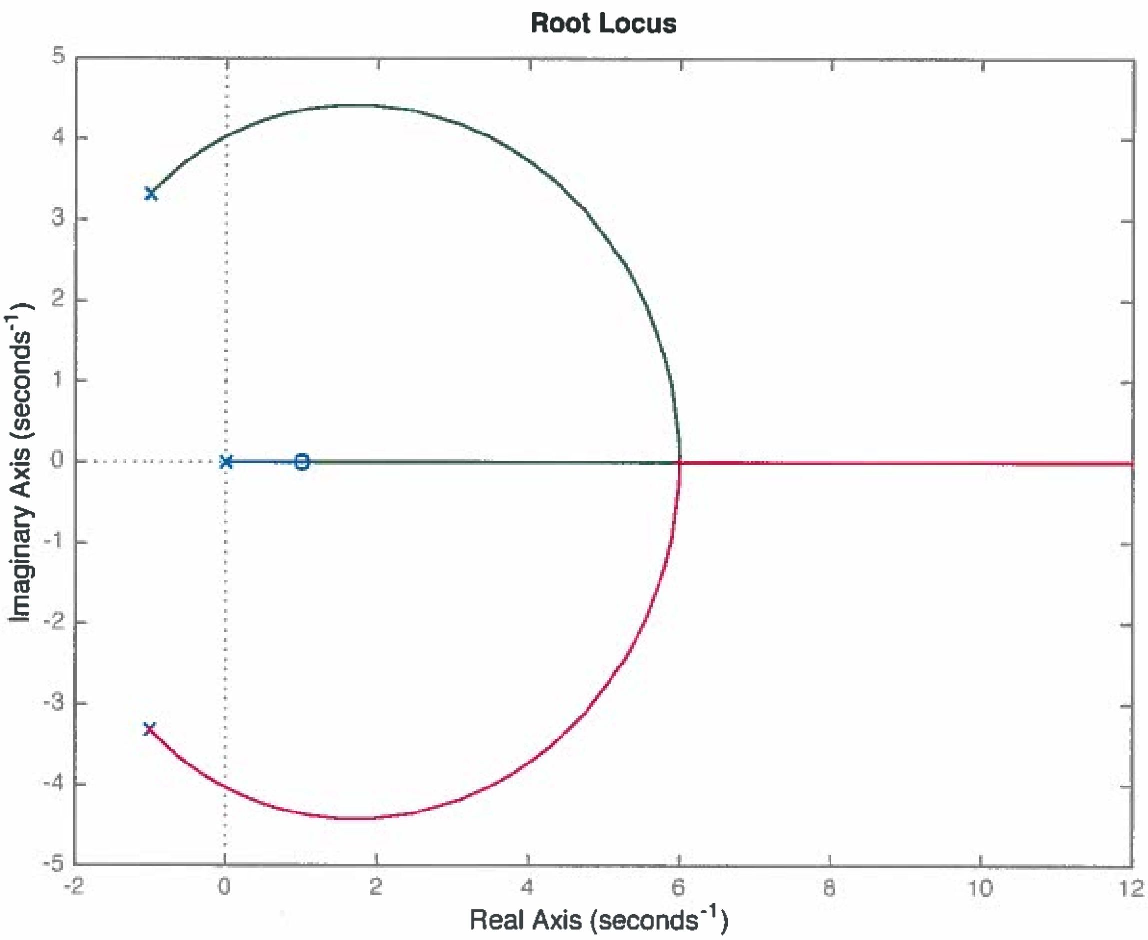
Double zero in $R(s)$ at $s=1$ must remain, hence there must be zero crossings in step response.



3(b)(i)


Root Locus
Imaginary Axis (seconds$^{-1}$)
Real Axis (seconds$^{-1}$)
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
0
1
2

3 (b)(i)


Root Locus
Imaginary Axis (seconds$^{-1}$)
Real Axis (seconds$^{-1}$)
5
4
3
2
1
0
-1
-2
-3
-4
-5
-2
0
2
4
6
8
10
12

**Q3. Non-minimum phase response, root-locus and two-degree-of-freedom design.**

15 attempts, mean 12.7/20 (63.7%), maximum 19, minimum 7.

A popular question with some very good answers, though a number of parts caused difficulties. Not many candidates got the root-locus correct for negative and positive $k$ in (b)(i). In (b)(iii) most had the general idea but few completed the two-degree-of-freedom design together with block diagram. Many candidates failed to draw the correct conclusions about the step response in (b)(iv).

3(b)(iv)