Prof R Sepulchre

4F2 Crib 2015

1(a) Consider a block diagonal structure

$$\Delta = \begin{bmatrix} \Delta_1 & 0 & & \\ 0 & \Delta_2 & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}$$

where each Δ_i is an (unstructured) H_∞ matrix. Then the structured singular value is defined as

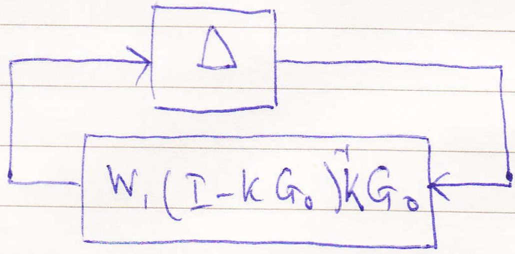
$$\mu(G) = \frac{1}{\min_{\Delta \text{ structured}} \{\bar{\sigma}(\Delta) \text{ s.t. } \det(I - G\Delta) = 0\}}$$

(b)(i) $u = KG_0(w+u)$ setting $d = 0$

$$\Rightarrow u = (I - KG_0)^{-1} KG_0 w$$

$$\Rightarrow z = W_1 (I - KG_0)^{-1} KG_0 w$$

Hence, stability of the feedback system is equivalent to K stabilising G_0 and the feedback system



being stable for all Δ with $\|\Delta\|_\infty < 1$. By the Small Gain Theorem this holds if and only if

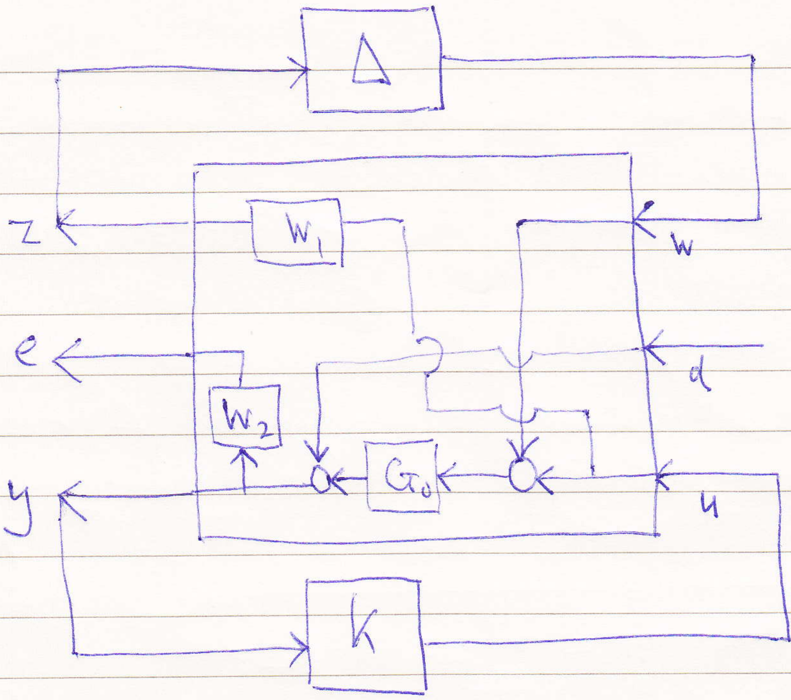
$$\|W_1 (I - KG_0)^{-1} KG_0\|_\infty \leq 1$$

Q1 Robustness and structured uncertainty

23 attempts, Average mark 12/20, Maximum 18, Minimum 1.

A number of candidates got mixed up between the structured singular value and the ordinary singular value of a matrix, which meant they failed to make progress with (a) and (b)(iv). In (b)(i) several candidates forgot that matrices don't commute. Generally parts (b)(ii) and (iii) were well done.

(b) (ii)



(iii)

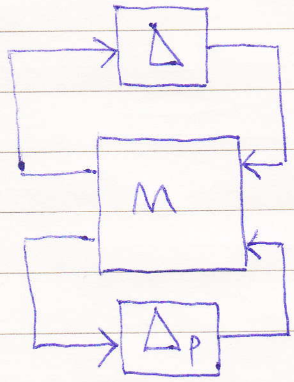
$$\begin{pmatrix} \hat{z} \\ \hat{e} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & G_0 \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{d} \\ \hat{r} \end{pmatrix}$$

and $$\begin{pmatrix} u \\ d \\ r \end{pmatrix} = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & I \end{pmatrix} \begin{pmatrix} w \\ d \\ u \end{pmatrix}$$

(where ^ denotes Laplace transform)

(true with or without ^)

(iv)

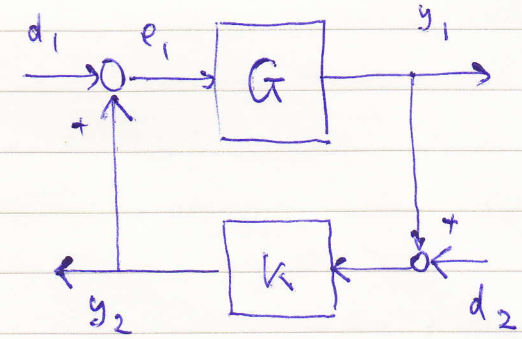


$\|T_{d\to e}\|_\infty \leq 1$ for all (unstructured) Δ with $\|\Delta\|_\infty \leq 1$

$$\iff \sup_\omega \mu(M) \leq 1$$

where the block structure is that of $\begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}$ with Δ_p unstructured.

2(a)(i)



The feedback system is defined to be internally stable if all transfer functions from d_1 and d_2 to e_1, e_2, y_1 and y_2 are in $\mathcal{H}_\infty$. Internal stability is equivalent to

$$\begin{pmatrix} I & -K \\ -G & I \end{pmatrix}^{-1} = \begin{pmatrix} (I-KG)^{-1} & K(I-GK)^{-1} \\ G(I-KG)^{-1} & (I-GK)^{-1} \end{pmatrix}$$

being in $\mathcal{H}_\infty$.

(ii) Suppose $\|GK\|_\infty < 1$. Then

$$\underline{\sigma}(I-GK) \geq 1-\bar{\sigma}(GK) > 0 \quad \text{for all } s \text{ with } Re(s) > 0$$

$$\Rightarrow (I-GK)^{-1} \text{ exists}$$

$$\Rightarrow GK(I-GK)^{-1},\ K(I-GK)^{-1},\ (I-GK)^{-1}G \text{ all stable}$$

$$\Rightarrow (I-KG)^{-1} = I + K(I-GK)^{-1}G \text{ is stable}$$

The case of $\|KG\|_\infty < 1$ is similar.

(b) Let $\sigma_1 \geq \sigma_2 \ldots \geq \sigma_r \geq 0$ be the singular values of M. Then σ_i^2 is an eigenvalue of M^*M and MM^* for any non-zero singular value

(c)(i)

$$\left| \frac{aj\omega}{(j\omega+a)^2} \right| = \frac{a\omega}{a^2+\omega^2}$$

achieves its maximum when $\omega = a$, which can be seen by considering the Bode plot or by differentiation. Hence

$$\left\| \frac{as}{(s+a)^2} \right\|_\infty = \frac{a^2}{2a^2} = \frac{1}{2}$$

(c)(ii) Note that $\left\|\frac{s}{s+a}\right\|_\infty = \left\|\frac{a}{s+a}\right\|_\infty = 1$.

$$\begin{pmatrix} 1 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix} = 101 \quad , \quad \begin{pmatrix} 9 & -1 \end{pmatrix}\begin{pmatrix} 9 \\ -1 \end{pmatrix} = 82$$

Hene $\|G\|_\infty = \sqrt{101}$ and $\|K\|_\infty = \sqrt{82}$.

For GK consider

$$M = \begin{pmatrix} 1 \\ 10 \end{pmatrix}\begin{pmatrix} 9 & -1 \end{pmatrix}$$

$$\Rightarrow \quad MM^* = \begin{pmatrix} 1 \\ 10 \end{pmatrix}\begin{pmatrix} 9 & -1 \end{pmatrix}\begin{pmatrix} 9 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 10 \end{pmatrix} = 82\begin{pmatrix} 1 & 10 \\ 10 & 100 \end{pmatrix}$$

$$\det\begin{pmatrix} 1-\lambda & 10 \\ 10 & 100-\lambda \end{pmatrix} = (\lambda-100)(\lambda-1)-100 = \lambda(\lambda-101)$$

Hence $\|GK\|_\infty = \sqrt{82}\sqrt{101}\,\frac{1}{2}$

$$KG = \begin{pmatrix} 9 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 10 \end{pmatrix}\frac{as}{(s+a)^2} = \frac{-as}{(s+a)^2} \Rightarrow \|KG\|_\infty = \frac{1}{2}$$

Since $\|KG\|_\infty < 1$ and G, K are stable, the feedback system is internally stable.

(iii) $\|KG\|_\infty = \|-KG\|_\infty = \|KGe^{-s\tau}\|_\infty$

Hence the internal stability is preserved with a negative feedback convention or with a time delay introduced.

Q2 H-infinity and internal stability

23 attempts, Average mark 13/20, Maximum 19, Minimum 5.

There were good attempts at this question by most candidates. The bookwork in (a)(ii) was often poor and there were very few really good attempts. A number of candidates made heavy weather of the computations in (c)(i) and (ii). Very few candidates used a Bode diagram to solve (c)(i). Nevertheless, most candidates knew where they were going with this question.

Q3. (a) equilibrium: $(\sqrt{x^2+h^2} - l_0)\, x = 0$

$\Rightarrow \quad x = 0 \quad$ or $\quad x = \pm\sqrt{l_0^2 - h^2}$

One equilibrium if $l_0 < h$ and three equilibria if $l_0 > h$

There will be a unique equilibrium provided that the spring is not compressed in the equilibrium configuration $x=0$.

(b) The potential energy of the NL spring is

$$H(x) = \int_0^x \underbrace{(\sqrt{s^2+h^2} - l_0)\, s}_{\geq 0 \text{ if } l_0 < h}\, ds$$

The total energy $V(x) = H(x) + \frac{m\dot{x}^2}{2}$ is a Lyapunov function (pos definite & radially unbounded). Its time derivative is

$$\dot{V} = \dot{x}\left(-d\dot{x} - (\sqrt{x^2+h^2} - l_0)\, x\right) + \frac{dH}{dx}\cdot\dot{x}$$

$$= -d\dot{x}^2 \leq 0$$

By LaSalle principle, all solutions converge to the largest invariant set where $\dot{x} = 0$, which is the equilibrium.

(c) State-space model is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}\left(-dx_2 - k\, h(x_1)\right)$$

with $\quad h(x_1) = \frac{dH}{dx_1} = (\sqrt{x_1^2+h^2} - l_0)\, x_1$

Jacobian linearization at $\bar{x} = (\bar{x}_1, \bar{x}_2)$:

$$J(\bar{x}) = \begin{pmatrix} 0 & 1 \\ -k\, h'(\bar{x}_1) & -\frac{d}{m} \end{pmatrix}$$

$$h'(\bar{x}_1) = \sqrt{\bar{x}_1^2 + h^2} - l_0 + \frac{\bar{x}_1^2}{\sqrt{\bar{x}_1^2 + h^2}}$$

$\bar{x}_1 = 0 \Rightarrow h'(\bar{x}_1) = h - l_0 < 0$

$\Rightarrow \left.\begin{array}{l} \det J(\bar{x}) < 0 \\ \text{trace } J(\bar{x}) < 0 \end{array}\right\} \Rightarrow \bar{x}$ saddle point

$\bar{x}_1 = \pm\sqrt{l_0^2 - h^2} \Rightarrow h'(\bar{x}_1) = \dfrac{l_0^2 - h^2}{\sqrt{l_0^2}} > 0$

$\Rightarrow \left.\begin{array}{l} \det J(\bar{x}) > 0 \\ \text{trace } J(\bar{x}) < 0 \end{array}\right\} \Rightarrow \bar{x}$ stable focus.

Q3 Lyapunov analysis of a nonlinear mass-spring-damper

22 attempts, Average mark 12/20, Maximum 20, Minimum 4.

There were good attempts at this question by most candidates. (a) was usually well done. For part (b), many candidates failed to find the expression of the total energy as a candidate Lyapunov function. For part (c), the main difficulty came from the calculation of the Jacobian matrix at different equilibria.

Q4. (a)

$$\begin{cases} \tau \dot{x}_1 = -x_1 + \dfrac{2}{1+x_m^k} \\ \tau \dot{x}_2 = -x_2 + x_1 \\ \vdots \\ \tau \dot{x}_m = -x_m + x_{m-1} \end{cases}$$

Equilibrium : $\bar{x}_1 = \bar{x}_2 = \cdots = \bar{x}_m = \dfrac{2}{1+\bar{x}_m^k}$

$\Rightarrow \quad \bar{x}_i = 1 \quad \forall i$

(b) $L(j\omega) = -\dfrac{\varphi'(\bar{x}_m)}{(\tau j\omega+1)^m} = \dfrac{k}{2(\tau j\omega+1)^m}$

Let $\omega^* > 0$ the smallest frequency for which $\operatorname{Im} L(j\omega^*) = 0$

$\Rightarrow$ phase $(\tau j\omega^* + 1) = \dfrac{\pi}{m}$

$\Rightarrow \quad \tau\omega^* = \tan\left(\dfrac{\pi}{m}\right)$

$\Rightarrow \quad |L(j\omega^*)| = \dfrac{k}{2(\tau^2\omega^2+1)^{m/2}}$

$$= \frac{k}{2\left(\frac{1}{\cos^2\left(\frac{\pi}{m}\right)}\right)^{m/2}} = \frac{k}{2}\left(\cos\left(\frac{\pi}{m}\right)\right)^m$$

$\Rightarrow$ The equilibrium is unstable if $|L(j\omega^*)| > 1$, that is

$$k > \frac{2}{\left(\cos\frac{\pi}{m}\right)^m}$$

For $n=2$, the equilibrium is stable for all $k>0$.

For $n=3$, the equilibrium is unstable for

$$k > \frac{2}{\left(\cos\frac{\pi}{3}\right)^3} = 16$$

(c) For $k \to +\infty$, $\varphi(x_n) \to 1 - \mathrm{sign}(x_n - 1)$

In the coordinates $z_i = x_i - 1$, the system is

$$\begin{cases} \tau \dot{z}_1 = -z_1 - \mathrm{sign}(z_n) \\ \tau \dot{z}_2 = -z_2 + z_1 \\ \vdots \\ \tau \dot{z}_n = -z_n + z_{n-1} \end{cases}$$

The nonlinearity $\mathrm{sign}(z_n)$ has the describing function

$$N(A,\omega) = \frac{4}{\pi A} \quad (\text{cf. handout})$$

Nyquist plot intersects $-\frac{\pi A}{4}$ at ω^*:

$$\begin{cases} A = \frac{4}{\pi}\left(\cos\left(\frac{\pi}{n}\right)\right)^n \\ \omega^* = \frac{1}{\tau}\tan\left(\frac{\pi}{n}\right) \end{cases}$$

Q4 Goodwin oscillator model and describing function analysis

5 attempts, Average mark 9/20, Maximum 16, Minimum 2.

Few students attempted this question, and the mean is the lowest. Reasoning was usually acceptable but students seem to have been destabilised by the fact that Goodwin model had not been discussed neither in lectures nor in example papers.