Module 4F2 - 2015-2016 - Cribs.

Question 1. *A popular and straightforward question, well-answered by most candidates. Almost no-one achieved a full mark on (b)(i), not recognizing the unboundedness of the norm for $k \geq 1$.*

(a)(i) $\gamma_1 = 2$ since

$$\sup_{1\leq\alpha\leq 2} \|G\|_\infty = \sup_{1\leq\alpha\leq 2} \sup_{\omega} |G(j\omega)| = \sup_{1\leq\alpha\leq 2} \sup_{\omega} \left(\frac{\alpha^2}{\omega^2+1}\right)^{\frac{1}{2}} = \sup_{1\leq\alpha\leq 2} \alpha\,.$$

(a)(ii) The small gain theorem requires $\|K\|_\infty < \frac{1}{2}$, therefore any gain $-\frac{1}{2} < k < \frac{1}{2}$ guarantees asymptotic stability of the closed loop system.

(b)(i) $\bar{z}(s) = kG_0(s)(\bar{w}(s) + \bar{z}(s)) = \frac{kG_0(s)}{I-kG_0(s)}\bar{w}(s) = \frac{k\frac{1}{s+1}}{I-k\frac{1}{s+1}}\bar{w}(s)$ thus

$$T_{w\to z}(s) = \frac{k}{s+1-k}\;.$$

For $k \geq 1$, $T_{w\to z}(s)$ has a pole at $k-1 \geq 0$ and its ∞-norm is unbounded. For $k < 1$,

$$\|T_{w\to z}(s)\|_\infty = \sup_{\omega} |T_{w\to z}(j\omega)| = \left(\frac{k^2}{\omega^2+(1-k)^2}\right)^{\frac{1}{2}} = \frac{k}{1-k}\;.$$

(b)(ii) Since $\|\Delta\|_\infty \leq 1$, the small gain theorem guarantee asymptotic stability of the closed loop if $\|T_{w\to z}(s)\|_\infty = |\frac{k}{1-k}| < 1$. Specifically, $\frac{k^2}{(1-k)^2} < 1$ if and only if $k^2 < (1-k)^2 = k^2 + 1 - 2k$, that is, $0 < 1-2k$. It follows that the closed loop is asymptotically stable for any $-\infty < k < \frac{1}{2}$.

(b)(iii) For $\|\Delta\|_\infty \leq \rho$, asymptotic stability of the closed loop is preserved if $\|T_{w\to z}(s)\|_\infty = |\frac{k}{1-k}| < \frac{1}{\rho}$. Note that $\lim_{|k|\to 0} \frac{k^2}{(1-k)^2} = 0$. Therefore, $\frac{k^2}{(1-k)^2} < \frac{1}{\rho^2}$ can be achieved, for any given ρ, by choosing a sufficiently small value for k.

Question 2. *There were good attempts at this question by most candidates. Some candidates did not consider the specific role of the weighting function W_2 in question (a). The bookwork in (b)(ii) was sometimes poor. (b)(iii) has been poorly addressed by most of the candidates, not taking advantage of standard arguments on sensitivity and complementary sensitivity functions. Most candidates knew where they were going with this question.*

(a)(i) By the small gain theorem, $K(s)$ guarantees robust stability for $G(s) = (I + 100\Delta(s))G_0(s) = (I + W_2(s)\Delta(s)W_2(s))G_0(s)$ if

$$\|T_{d\to y_0}\|_\infty < \frac{1}{\|W_2 \Delta W_2\|_\infty} \ .$$

This is equivalent to

$$\|T_{w_2\to z_2}\|_\infty < \frac{1}{\|\Delta\|_\infty} \ .$$

By assumption, $\|T_{w_2\to z_2}\|_\infty < 1 = \frac{1}{\|\Delta\|_\infty}$, as required.

(a)(ii) For the first bound,

$$\begin{aligned} 1 > \|T_{w_2\to z_2}\|_\infty &= \sup_\omega \bar\sigma(T_{w_2\to z_2}(j\omega)) = \sup_\omega \bar\sigma(W_2(j\omega)T_{d\to y_0}(j\omega)W_2(j\omega)) \\ &= 100 \sup_\omega \bar\sigma(T_{d\to y_0}(j\omega)) = 100\|T_{d\to y_0}\|_\infty \ . \end{aligned}$$

For the second bound, note that $T_{r\to e}(s) - T_{d\to y_0}(s) = 1$. Thus, $1 = \bar\sigma(T_{r\to e}(j\omega) - T_{d\to y_0}(j\omega)) \leq \bar\sigma(T_{r\to e}(j\omega)) + \bar\sigma(T_{d\to y_0}(j\omega))$ therefore $\bar\sigma(T_{r\to e}(j\omega)) \geq 1 - \bar\sigma(T_{d\to y_0}(j\omega))$. It follows that

$$\|T_{r\to e}\|_\infty = \sup_\omega \bar\sigma(T_{r\to e}(j\omega)) \geq 1 - \sup_\omega \bar\sigma(T_{d\to y_0}(j\omega)) = 1 - \|T_{d\to y_0}\|_\infty > \frac{99}{100} \ .$$

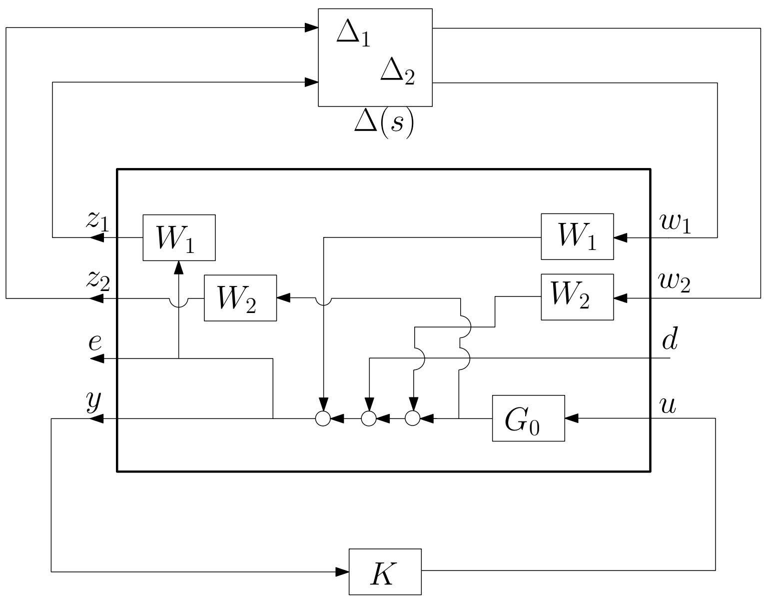


Figure 1: (b)(i)

(b)(ii) The robust performance problem addresses the question of what is the largest possible gain from the disturbances to the errors in the presence of the uncertainties Δ. Using the form of a generalized plant P and an extended set of perturbations, the robust performance problem can be addressed as a robust stability problem. For this reason, robust performance can be approached using the same techniques as robust stability, for example by a design based on structured singular value. A detailed characterization is in Section 4.4.3 of the handouts.

(b)(iii) Consider $\frac{1}{\alpha^2} + \frac{1}{\beta^2} \leq 1$. Note that

$$\|T_{w_1 \to z_1}\|_\infty = \|W_1 T_{r \to e} W_1\|_\infty = \alpha^2 \|T_{r \to e}\|_\infty \; ;$$
$$\|T_{w_2 \to z_2}\|_\infty = \|W_2 T_{d \to y_0} W_2\|_\infty = \beta^2 \|T_{d \to y_0}\|_\infty \; .$$

Since $T_{r \to e}(s) - T_{d \to y_0}(s) = 1$,

$$1 = \|T_{r \to e} - T_{d \to y_0}\|_\infty \leq \|T_{r \to e}\|_\infty + \|T_{d \to y_0}\|_\infty \leq \frac{1}{\alpha^2}\|T_{w_1 \to z_1}\|_\infty + \frac{1}{\beta^2}\|T_{w_2 \to z_2}\|_\infty \; . \tag{1}$$

Suppose now that some controller K guarantees $\|T_{w_1 \to z_1}\|_\infty < 1$ and $\|T_{w_2 \to z_2}\|_\infty < 1$. Then, from (1) we reach the contradiction $1 < \frac{1}{\alpha^2} + \frac{1}{\beta^2} \leq 1$ since

$$\frac{1}{\alpha^2}\|T_{w_1 \to z_1}\|_\infty + \frac{1}{\beta^2}\|T_{w_2 \to z_2}\|_\infty < \frac{1}{\alpha^2} + \frac{1}{\beta^2} \leq 1 \; .$$

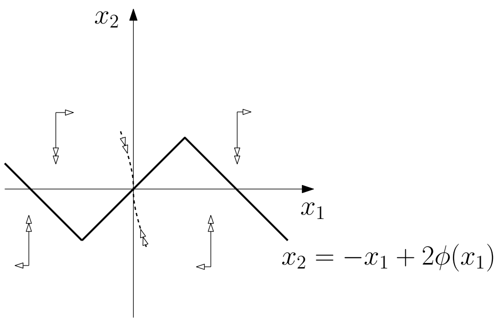
Question 3. *The least popular question, with the lowest mean. Reasoning simultaneously in the state-space domain and in the frequency domain appears to be the dominant obstacle.*

(a) The state-space model of the feedback system is $\dot{x} = -x \pm 2\phi(x)$. Equilibria are defined by $x = \pm 2\phi(x)$. The negative feedback system has one stable equilibrium at $x = 0$ (the linearized behavior is $\delta\dot{x} = -3\delta x$). The positive feedback system has three equilibria: $-2, 0, 2$. The linearization is $\delta\dot{x} = -\delta x$ when the saturation is active and $\delta\dot{x} = +\delta x$ when the saturation is non active. Therefore the positive feedback system is a bistable system. The unstable equilibrium divides the state-space into two disjoint basins of attraction.

(b) The state-space model of the feedback system is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \epsilon\dot{x}_2 &= -x_2 - x_1 + 2\phi(x_1) \end{aligned} \tag{2}$$

The system is a fast-slow system because of the small parameter ϵ. Trajectories converge in the fast time scale towards the isocline $x_2 = x_1 + 2\phi(x_1)$. In the slow time scale, they slide along the isocline to converge either to the equilibrium $x_1 = 2$ or to the equilibrium $x_1 = -2$. The separatrix between the two basins of attraction is the stable manifold of the saddle point $x_1 = 0$. The separatrix is close to the vertical axis $x_1 = 0$ when ϵ is small.



(c) Choose $H(s)$ for the equilibrium at zero to be unstable. For instance, $H(s) = \frac{K}{(s+1)^3}$ will make the origin unstable if K is large enough, say $K = 2$. Because the linear system is stable, a bounded input produces a bounded output. Because of the saturation, the input of the linear system is always bounded. Therefore trajectories cannot grow unbounded. Bounded trajectories around an unstable equilibrium suggest an oscillatory behavior. The frequency and the amplitude of the oscillation can be predicted by describing function analysis. Because ϕ is a odd nonlinearity, its describing function is a real function of ω, therefore the frequency of the oscillation is provided by the intersection of $H(j\omega)$ with the negative real axis, that is, for $\omega = \frac{1}{2}$. The amplitude is the oscillation is the solution of the equation $H(j\frac{1}{2})N(A, \frac{1}{2}) + 1 = 0$.

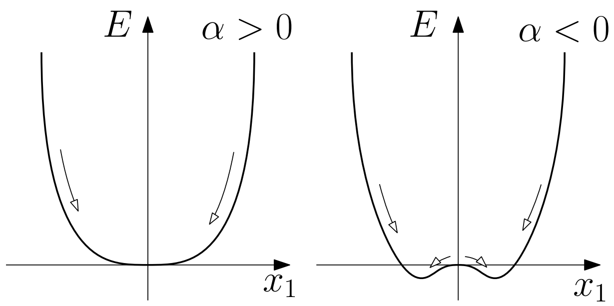
(d) Equilibria only depend on the static gain $H(0)$ and conclusions about their existence are the same as in (a) if $H(0) > 1$. Stability of the equilibria where the saturation is active is also unchanged, since the linerization at those equilibria reduces to the stable (open-loop) transfer function $H(s)$. Stability of the zero equilibrium is the stability of the linear system $H(s)$ in a unity feedback configuration. Here the conclusions for the transfer function in (a) can be different than the conclusions for an arbitrary stable transfer function (as shown for instance by the example in (c)).

Question 4 *A popular question, generally well answered. Marks were mostly lost in d(i) .*

(a)

$$\dot{V} = \frac{\partial E}{\partial x}\dot{x} + \dot{x}(-\frac{\partial E}{\partial x} - k\dot{x}) = -k\dot{x}^2 \leq 0$$

(b) The potential is a single well when $\alpha \geq 0$ and a double well when $\alpha > 0$. The Duffing system describes the behavior of a ball rolling in the well with a friction coefficient $k > 0$ and a horizontal external forcing of the well by u.



(c) The equilibrium is stable when $k = 0$ because V is positive definite and non increasing along solutions. For $k > 0$, Lyapunov analysis implies convergence of all solutions to the set $\dot{x} = 0$. The largest *invariant* set in that set is the equilibrium. Therefore the equilibrium is stable. Because V is proper, the equilibrium is *globally* asymptotically stable, that is, the basin of attraction is the entire state-space.

(d)(i) With a nonzero force u, we have

$$\dot{V} = -k\dot{x}^2 + u\dot{x} \leq u\dot{x}$$

This means that the system is passive from the force u to the output $y = \dot{x}$, with storage V and supply rate uy. The storage is the internal energy of the mechanical system, whereas the supply rate is the mechanical power.

(d)(ii) The transfer function $H(s)$ is strictly positive real, i.e. the real part of $H(j\omega)$ is strictly positive for any ω. The passivity theorem states that the feedback interconnection of a strictly passive and a passive system defines a strictly passive system. The implication is that the feedback controller $u = -k_p\dot{x} - k_i x$ makes the feedback system stable for any choice of $k_p > 0$ and $k_i > 0$.

(e)(i) The linearisation is $\delta\ddot{x} - \alpha\delta x + \delta\dot{x} = 0$. The product of the two roots of the characteristic equation is $-\alpha < 0$, which means that the two eigenvalues of the Jacobian matrix are real and of opposite sign, i.e. the equilibrium is a saddle point.

(e)(ii) The energy looks indeed as a saddle near $(x, \dot{x}) = (0, 0)$.

(e)(iii) For small damping, the behavior is the behavior of a ball rolling with small friction in a double well. If the ball starts close to the saddle, it returns close to the saddle and only needs a little perturbation to fall down on either side of the double well. This means that under small forcing, the long-term behavior is hard to predict even if the forcing is deterministic (for instance harmonic forcing).