4F2 cribs, 2018

Q1 Internal stability, small gain theorem, block diagrams

23 attempts, Average mark 13.9, Maximum 19, Minimum 8.

A popular question, well-answered by most candidates. Part (a), (b), and (d) were well answered, in general. A few mistakes in the analysis of controller K3 in part (c). Difficulties in handling norms correctly in Part (e).

(a) From the block diagram we have $e_1 = d_1 + Ke_2$ and $e_2 = d_2 + Ge_1$ therefore

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} T_{d_1 \to e_1} & T_{d_2 \to e_1} \\ T_{d_2 \to e_1} & T_{d_2 \to e_2} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

where T_{d_i,e_j} is the transfer function from the input d_i to the output e_j. Specifically

$$\begin{aligned} e_1 &= d_1 + Kd_2 + KGe_1 = \underbrace{(I-KG)^{-1}}_{T_{d_1 \to e_1}} d_1 + \underbrace{(I-KG)^{-1}K}_{T_{d_2 \to e_1}} d_2 \\ e_2 &= d_2 + Gd_1 + GKe_2 = \underbrace{(I-GK)^{-1}G}_{T_{d_1 \to e_2}} d_1 + \underbrace{(I-GK)^{-1}}_{T_{d_2 \to e_2}} d_2 \end{aligned}$$

Using the suggested identities, note that $T_{d_2 \to e_1} = K(I - GK)^{-1}$ and $T_{d_1 \to e_2} = G(I - KG)^{-1}$, which conclude the answer.

(b) We show that each component of the matrix is bounded. Define $a = \| G \|_\infty$ and $b = \| K \|_\infty$, both well defined since G and K are in $\mathcal{H}_\infty$. Define $c = \| KG \|_\infty < 1$.

$$\begin{aligned} \| (I-KG)^{-1} \|_\infty &\leq \frac{1}{1 - \| KG \|_\infty} = \frac{1}{1-c} \\ \| G(I-KG)^{-1} \|_\infty &\leq \frac{\| G \|_\infty}{1 - \| KG \|_\infty} = \frac{a}{1-c} \\ \| K(I-GK)^{-1} \|_\infty &= \| (I-KG)^{-1}K \|_\infty \leq \frac{\| K \|_\infty}{1 - \| KG \|_\infty} = \frac{b}{1-c} \\ \| (I-GK)^{-1} \|_\infty &= \| I + G(I-KG)^{-1}K \|_\infty \leq 1 + \frac{\| G \|_\infty \| K \|_\infty}{1 - \| KG \|_\infty} = 1 + \frac{ab}{1-c} \end{aligned}$$

The closed loop is internally stable since all transfer functions from d_1, d_2 to e_1, e_2, y_1, y_2 are in $\mathcal{H}_\infty$. This is equivalent to show that $\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1}$ is in $\mathcal{H}_\infty$.

(c) K_1 guarantees nominal closed loop stability since $\| K_1 G \|_\infty = \| G \|_\infty < 1$. K_3 guarantees nominal closed loop stability since $\| K_3 G \|_\infty = \max_\omega |K_3(j\omega)G(j\omega)|$ and at each frequency $|K_3(j\omega)G(j\omega)| \leq |K_3(j\omega)||G(j\omega)| < 1$.

The stability of the closed loop system with the controller $K = K_2$ cannot be determined from the Bode diagrams. We have $\| K_2 G \|_\infty > 1$ but this does not imply instability of the closed loop, in general.

(d) Additive uncertainties act on the closed loop system as shown in Fig. 1 below. By the small gain theorem, the closed loop is stable against any perturbation Δ if the transfer function $T_{w\to z}$ from w to z satisfies

$$\| T_{w\to z} \|_\infty = \| (I - KG)^{-1}K \|_\infty < \frac{1}{\| \Delta \|_\infty} .$$

(e) From (c), recall that $K = K_1$ and $K = K_3$ guarantee $\max_\omega |K(j\omega)G(j\omega)| < 1$. Consider

$$\| (I - KG)^{-1}K \|_\infty \le \max_\omega \frac{|K(j\omega)|}{1 - |K(j\omega)G(j\omega)|} \le \frac{\| K \|_\infty}{1 - \| KG \|_\infty} .$$

From (d), robust stability is thus guaranteed if

$$\frac{\| K \|_\infty}{1 - \| KG \|_\infty} < \frac{1}{\| \Delta \|_\infty} \quad \text{that is} \quad \| \Delta \|_\infty < \frac{1 - \| KG \|_\infty}{\| K \|_\infty} .$$

It follows that

$$b = \frac{1 - \| KG \|_\infty}{\| K \|_\infty} .$$

We now use the Bode diagrams.

For $K = K_1$, $\| K_1 \|_\infty = 1$ and $\| K_1G \|_\infty \le -6$ dB $\simeq 0.5012$. Thus,

$$b = 1 - 0.5012 = 0.4988 .$$

For $K = K_3$, $\| K_3 \|_\infty \le 8$ dB $\simeq 2.5119$ and $\| K_3G \|_\infty \le -6$ dB $\simeq 0.5012$. Thus,

$$b = \frac{1 - 0.5012}{2.5119} = 0.1986 .$$

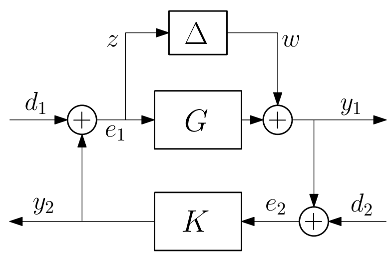


Figure 1: Closed loop with additive uncertainties.

Q2 Robust stability, performances, uncertainties

14 attempts, Average mark 11.9/20, Maximum 18, Minimum 5.

An unpopular question. Part (a), (b) and (e) were well answered, with minor issues in computations. Not many students got the right range of stabilizing gains in Part (c). Only a few students did a complete analysis in Part (d).

(a) We solve

$$A_0(s)(1 + \bar{\Delta}(s)) = \frac{1}{s+2}(1 + \bar{\Delta}(s)) = \frac{1}{s+a} \qquad 1 < a < 2$$

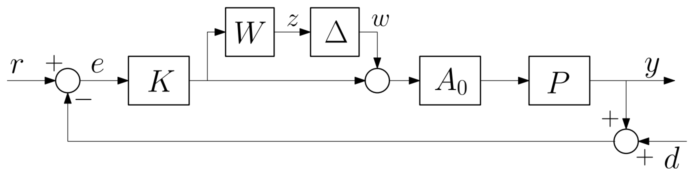
which gives

$$\bar{\Delta}(s) \;=\; (s+2)\left(\frac{1}{s+a} - \frac{1}{s+2}\right) \;=\; \frac{s+2}{s+a} - 1 \;=\; \frac{2-a}{s+a} = \underbrace{\frac{2}{s+2}}_{W(s)}\underbrace{\frac{(2-a)(s+2)}{2(s+a)}}_{\Delta(s)} \;.$$

Furthermore,

$$\| \Delta \|_\infty = \frac{1}{2}\sup_{\omega, 1\le a\le 2}(2-a)\sqrt{\frac{\omega^2+2}{\omega^2+a}} \;<\; \frac{1}{2}\sup_{\omega}\sqrt{\frac{\omega^2+2}{\omega^2+1}} \;=\; 1 \;.$$

Diagram of the closed loop with multiplicative uncertainties on the actuator input:



(b) Compute the nominal transfer functions

$$e = r - P(s)A_0(s)K(s)e = (I + P(s)A_0(s)K(s))^{-1}r = \frac{r}{1+\frac{k}{s(s+2)}} = \frac{s(s+2)}{s^2+2s+k}r$$

and

$$y = -P(s)A_0(s)K(s)(d+y) = -(I+P(s)A_0(s)K(s))^{-1}P(s)A_0(s)K(s)d = -\frac{k}{s^2+2s+k}d$$

that is

$$T_{r\to e}(s) = \frac{s(s+2)}{s^2+2s+k} \qquad T_{d\to y}(s) = \frac{-k}{s^2+2s+k} \;.$$

Both transfer functions have poles at

$$p_{1,2} = \frac{-2 \pm 2\sqrt{1-k}}{2}$$

whose real part is always smaller than 0 for $k > 0$. Therefore, both transfer functions are proper and have no poles in RHP or on the imaginary axis, thus both transfer functions are in $\mathcal{H}_\infty$.

(c) Robust stability holds for any gain $k > 0$.

Consider the block diagram above (part (a)). From part (a), $W(s) = \frac{2}{s+2}$ and $\| \Delta(s) \|_\infty < 1$, thus $\frac{1}{\|\Delta(s)\|_\infty} > 1$. By the small gain theorem, robust stability is guaranteed if

$$\| T_{w\to z} \|_\infty \le 1$$

where $T_{w\to z}$ is the transfer function from w to z. We have

$$\begin{aligned} z \;&=\; W(s)K(s)e = \frac{2k}{s+2} \\ e \;&=\; -P(s)A_0(s)(w + K(s)e) = -(I + P(s)A_0(s)K(s))^{-1}P(s)A_0(s)w = \frac{-1}{s^2+2s+k}w \end{aligned}$$

therefore

$$T_{w\to z}(s) = \frac{-2k}{(s^2+2s+k)(s+2)} \ .$$

It follows that

$$|T_{w\to z}(j\omega)| = \sqrt{\frac{4k^2}{((k-\omega^2)^2+4\omega^2)\,(\omega^2+4)}}$$

whose maximum $\| T_{w\to z} \|_\infty$ is attained either at $\omega = 0$ or at the peak of the second order network $\frac{1}{s^2+2s+k}$, given by

$$\frac{\partial}{\partial \omega^2}\left((k-\omega^2)^2+4\omega^2\right) = -2(k-\omega^2)+4 = 0 \quad \rightarrow \quad \omega = \sqrt{k-2}$$

(the peak exists for $k > 2$). For all $k \geq 0$,

$$|T(j0)| = \frac{2k}{2k} = 1$$

and, for $k \geq 2$,

$$|T(j\sqrt{k-2})| = \sqrt{\frac{4k^2}{4(k-1)(k+2)}} = \sqrt{\frac{4k^2}{4(k^2+k-2)}} \leq 1 \ ,$$

which shows that $\| T_{w\to z} \|_\infty \leq 1$.

(d) The tracking performance at frequency ω is given by

$$|T_{r\to e}(j\omega)| = \left|\frac{-\omega^2+2j\omega}{-\omega^2+2j\omega+k}\right| = \sqrt{\frac{-\omega^4+4\omega^2}{(k-\omega^2)^2+4\omega^2}} \ .$$

$|T_{r\to e}(j\omega)|$ gets smaller for k (sufficiently) large since $\lim\limits_{k\to\infty} |T_{r\to e}(j\omega)| = 0$, improving tracking at frequency ω.

By the small gain theorem, robust stability to multiplicative uncertainties $\bar{\Delta}(s)$ on the plant output is guaranteed if $\| T_{d\to y} \|_\infty < 1/ \| \bar{\Delta} \|_\infty$. Necessarily then

$$|T_{d\to y}(j\omega)| < 1/|\bar{\Delta}(j\omega)| \qquad \forall \omega \ .$$

However, by construction, $|T_{r\to e}(j\omega) + T_{d\to y}(j\omega)| = 1$, that is, for large k we have $|T_{d\to y}(j\omega)| \simeq 1$. Thus, stability in the presence of uncertainties can be achieved only if $\| \bar{\Delta} \|_\infty < 1$.

(e) The robust performance problem addresses the question of what is the largest possible gain from the disturbances to the errors in the presence of the uncertainties Δ. The generalized plant in Fig. 2 is a representation of the system based on three main input/output pairs: (u, y) for control action, (d, e) for performance characterization, and (w, z) for robustness. Using the form of a generalized plant and an extended set of perturbations Δ, the robust performance problem can be addressed as a robust stability problem, for example by a design based on structured singular value. Please refer to Section 4.4.3 of the handouts.

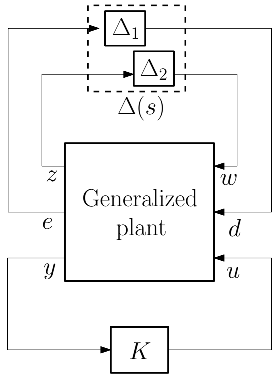


Figure 2: Generalized plant

Q3 Lyapunov stability

23 attempts, Average mark 15.4/20, Maximum 20, Minimum 7.

The definitions in the first part of the question were generally well answered though many students failed to note that an asymptotically stable equilibrium point is also stable. In part (c) many students failed to note that LaSalle?s theorem would be needed to characterize the asymptotic behaviour of the system.

(a) (i) An equilibrium point x_e is stable if $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\|x(0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \epsilon\ \forall t > 0$.

(ii) An equilibrium point x_e is asymptotically stable if it is stable and $\exists M > 0$ s.t. $\|x(0) - x_e\| < M \Rightarrow x(t) \to x_e$ as $t \to \infty$.

(iii) An equilibrium point x_e is globally asymptotically stable if it is asymptotically stable and M in (ii) can be chosen arbitrarily large.

(iv) A set D is the domain of attraction of x_e if $x(0) \in D \Rightarrow x(t) \to x_e$ as $t \to \infty$.

(b) Lyapunov's indirect method. An equilibirium x_e is

(i) asymptotically stable if the equilibrium $\delta x = 0$ of its linearization is asymptotically stable,

(ii) unstable if the equilibrium $\delta x = 0$ of its linearization is unstable.

Limitations. Inconclusive when $\delta x = 0$ in the linearization is stable, but not asymptotically stable. Cannot be used to prove global asymptotic stability.

(c) (i) $x = y = 0$ is an equilibrium point.

$$
\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i + \sum_{j=1}^{m} \frac{\partial V}{\partial y_j}\dot{y}_j \\
&= \sum_{i=1}^{n} g_i(x_i)\dot{x}_i + \sum_{j=1}^{m} f_j(y_j)\dot{y}_j \\
&= -\sum_{i=1}^{n} \alpha_i g_i(x_i)x_i - \sum_{i=1}^{n}\left[g_i(x_i)\sum_{j=1}^{m} f_j(y_j)\right] - \sum_{j=1}^{m} \beta_j f_j(y_j)y_j + \sum_{j=1}^{m}\left[f_j(y_j)\sum_{i=1}^{n} g_i(x_i)\right] \\
&= -\sum_{i=1}^{n} \alpha_i g_i(x_i)x_i - \sum_{j=1}^{m} \beta_j f_j(y_j)y_j \\
&\leq 0
\end{aligned}
$$

$\dot{V} = 0$ implies $y_j = 0$ since $\beta_j > 0$ but not necessarily $x_i = 0$ since α_i can be zero. So we apply Lasalle's invariance principle to characterize the asymptotic behaviour of the system.

The set $S := \{(x, y) : V(x, , y) \leq c\}$ for some $c > 0$ is a compact invariant set. $\dot{V} = 0 \Rightarrow y_j = 0\ \forall j$ and $x_i = 0$ for i s.t. $\alpha_i > 0$. If $\alpha_k = 0$ for some k and $\alpha_i > 0$ for $i \neq k$, then $x_k(t) \neq 0$ implies $\dot{y}_j(t) \neq 0$, and hence there exists $\tau > t$ s.t. $y_j(\tau) \neq 0$. So the largest invariant set for which $\dot{V} = 0$ is the origin, and for any initial condition in S we have convergence to the origin.

If $\alpha_i = 0$ for more than one values of i then denoting by Z the set of those values of i we have that $\dot{V} = 0$ implies that $x_i = 0$ for $i \notin Z$, $y_j = 0\ \forall j$ and $\sum_{i \in Z} g_i(x_i) = 0$. These are equilibrium points, hence for any initial condition in S we have convergence to the set of equilibrium points that lie in S.

(ii) $V(x, y)$ satisfies $V(x, y) \to \infty$ as $\|x\ y\| \to \infty$ since f_j, g_i are non decreasing functions. Hence the set S in (c)(i) can be chosen arbitrarily large, so the origin is globally asymptotically stable.

Q4 Describing function

21 attempts, Average mark 15.9/20, Maximum 20, Minimum 8.

The question was attempted by most students and was generally well answered. Most students obtained the correct answer for part (a). Some found parts (b) and (d) more challenging.

(a)

$$N(E) = \frac{U_1 + jV_1}{E}$$

The function f is odd so $V_1 = 0$.

$$U_1 = \frac{1}{\pi} \int_0^{2\pi} f(E \sin\theta) \sin\theta d\theta$$

If $E \leq \delta$ then $f(E \sin\theta) = 0$ so $U_1 = 0$. If $E > \delta$ then

$$\begin{aligned}
U_1 &= \frac{4}{\pi} \int_0^{\pi/2} f(E \sin\theta) \sin\theta d\theta \\
&= \frac{4}{\pi} \int_{\sin^{-1}(\delta/E)}^{\pi/2} E \sin^2\theta d\theta \\
&= \frac{4}{\pi} \int_{\sin^{-1}(\delta/E)}^{\pi/2} \frac{E}{2}(1 - \cos(2\theta)) d\theta \\
&= \frac{2E}{\pi} \left[\theta - \frac{\sin(2\theta)}{2}\right]_{\sin^{-1}(\delta/E)}^{\pi/2} \\
&= \frac{2E}{\pi} \left[\frac{\pi}{2} - \sin^{-1}(\delta/E) + \frac{1}{2} \sin\left(2 \sin^{-1}(\delta/E)\right)\right] \\
&= E - \frac{2E}{\pi} \left[\sin^{-1}(\delta/E) + \frac{\delta}{E} \sqrt{1 - \left(\frac{\delta}{E}\right)^2}\right]
\end{aligned}$$

(b)

$$G(j\omega) = \frac{\alpha}{(2j\omega+1)^2} = \frac{\alpha(1-2j\omega)^2}{(4\omega^2+1)^2} = \frac{\alpha(1-4\omega^2-4j\omega)}{(4\omega^2+1)^2}$$

$$\Re[G(j\omega)] = \frac{\alpha(1-4\omega^2)}{(4\omega^2+1)^2}$$

$$\frac{d[\Re[G(j\omega)]}{d\omega^2} = \alpha\frac{-4(4\omega^2+1)^2 - 8(4\omega^2+1)(1-4\omega^2)}{(4\omega^2+1)^4} = 0$$
$$\Rightarrow (4\omega^2+1)(-4\omega^2+3) = 0 \Rightarrow \omega^2 = 3/4$$

So

$$\Re[G(j\omega)] = -\frac{\alpha}{8}$$

The nonlinearity $f(e)$ is sector bounded with gain ψ satisfying $0 \le \psi \le 1$. So from the circle criterion we need for stability $\Re[G(j\omega)] > -1 \Rightarrow 0 < \alpha < 8$.

The circle criterion is only a sufficient condition for stability so this does not imply the feedback system is unstable for $\alpha > 8$.

(c) Noting that $f(e) \le e$ we have $0 < N(E) \le 1$. For harmonic balance we need

$$-\frac{1}{N(E)} = G(j\omega)$$

$$\Im[G(j\omega)] = \frac{-4\alpha\omega}{(4\omega^2+1)^2} = 0 \Rightarrow \omega = 0, \omega \to \infty$$

$$\omega = 0 \Rightarrow G(j\omega) = \alpha$$
$$\omega \to \infty \Rightarrow G(j\omega) \to 0$$

Hence harmonic balance never achieved for $\alpha > 0$, so no limit cycle predicted.

(d) For $\delta = 0$ $f(e) = e$, i.e. a static linear gain of 1. Hence need to find the value of β s.t. $\Re[G(j\omega)] = -1$ for some ω.

$$G(j\omega) = \frac{\beta}{(j\omega+1)^3} = \frac{\beta}{(\omega^2+1)^{\frac{3}{2}}} e^{-3j\tan^{-1}\omega}$$

$$\Im[G(j\omega)] = 0 \Rightarrow -3\tan^{-1}\omega = -\pi \Rightarrow \omega = \tan(\pi/3) = \sqrt{3}$$

At this frequency

$$\Re[G(j\omega)] = -|G(j\omega| = -\beta/8$$

So for oscillation need $\beta = 8$.