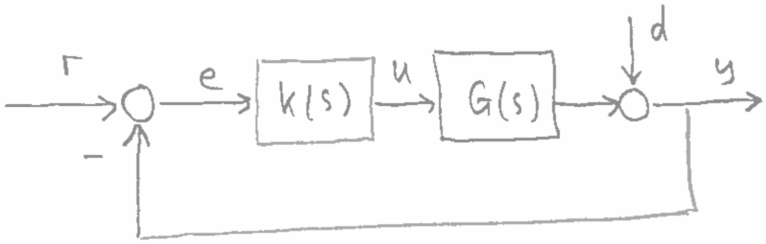
4F2 crib 2019

1. (a)(i)



$$\hat{y} = GK(\hat{r} - \hat{y}) + \hat{d}$$

$$\Rightarrow (I + GK)\hat{y} = GK\hat{r} + \hat{d}$$

$$\Rightarrow \hat{y} = (I + GK)^{-1}\hat{d} + (I + GK)^{-1}GK\hat{r}$$

(ii) $S + T = I \Rightarrow \underline{\sigma}(I) \leq \bar{\sigma}(S) + \underline{\sigma}(T)$

$$\Rightarrow \underline{\sigma}(T) \geq 1 - \bar{\sigma}(S)$$

$$\bar{\sigma}((I+L)^{-1}) = (\underline{\sigma}(I+L))^{-1}$$

$$\underline{\sigma}(L) = \underline{\sigma}(I + L - I) \leq \underline{\sigma}(I+L) + \bar{\sigma}(I)$$

$$\Rightarrow \underline{\sigma}(I+L) \geq \underline{\sigma}(L) - 1$$

(iii) (1) means T can't be small when S is small

(2) " Large $\underline{\sigma}(L)$ can make S small $\Rightarrow$ good disturbance rejection and tracking. Note T small is needed at frequencies where there is much sensor noise and plant uncertainty

(b)(i)

$$GK_1 = \frac{1}{(s+1)(0.25s+1)}\begin{bmatrix} e^{-0.2s} & 3 \\ 2.5e^{-0.5s} & -2.5e^{-0.3s} \end{bmatrix}\begin{bmatrix} 0.25 & 0.3 \\ 0.25e^{-0.2s} & -0.1e^{-0.2s} \end{bmatrix}$$

$$= \frac{1}{(s+1)(0.25s+1)}\begin{bmatrix} e^{-0.2s} & 0 \\ 0 & e^{-0.5s} \end{bmatrix}$$

(ii) $\bar{\sigma}(L) = \underline{\sigma}(L) = \left|\frac{1}{j\omega(j\omega+1)2}\right| = \frac{1}{2\omega(1+\omega^2)^{1/2}}$

$\bar{\sigma}(S) \leq 0.5$ requires $\underline{\sigma}(L) \geq 3 \Rightarrow 1 \geq 6\omega(1+\omega^2)^{1/2}$

$\frac{1}{36} = \omega^2(1+\omega^2) \Rightarrow \omega^2 = \frac{-1 \pm \sqrt{1+\frac{1}{9}}}{2} \Rightarrow \omega = 0.164$ rad/sec

Below this frequency $\bar{\sigma}(S) \leq 0.5$

(iii) $\bar{\sigma}(L) \to 0$ as $\omega \to \infty$ so $\bar{\sigma}(T) \to 0$ as $\omega \to \infty$

2 (a) $\left.\begin{array}{l}\hat{u} = \hat{r} - k\hat{e} \\ \hat{e} = \hat{w} - G\hat{u}\end{array}\right\} \Rightarrow \hat{u} = \hat{r} - k(\hat{w} - G\hat{u})$

$\Rightarrow \hat{u} = (I - kG)^{-1}(I \;\; -k)\begin{pmatrix}\hat{r}\\ \hat{w}\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}\hat{u}\\ \hat{y}\end{pmatrix} = H\begin{pmatrix}\hat{r}\\ \hat{w}\end{pmatrix}$$

(b) $(I-kG)^{-1}$, $G(I-kG)^{-1}$, $(I-kG)^{-1}k$ and $G(I-kG)^{-1}k$ are all elements of H so keeping a reasonable bound on $\|H\|_\infty$ is needed to ensure disturbances are not greatly amplified. $b_{G,k}$ is also the robustness measure for uncertainty in the gap metric, so keeping it as large as possible is good for robustness.

(c)
$$H(j\omega)\begin{pmatrix}\hat{u}_0\\ \hat{y}_0\end{pmatrix} = \begin{pmatrix}I\\ G(j\omega)\end{pmatrix}(I - k(j\omega)G(j\omega))^{-1}(I - k(j\omega)G(j\omega))\hat{u}_0$$
$$= \begin{pmatrix}\hat{u}_0\\ G(j\omega)\hat{u}_0\end{pmatrix} = \begin{pmatrix}\hat{u}_0\\ \hat{y}_0\end{pmatrix}$$

$r = u_0,\ w = y_0 \Rightarrow u = u_0,\ y = y_0,\ e = 0,\ v = 0$

(d) Since $\|H\|_\infty = \sup_{\left\|\binom{\hat{u}}{\hat{y}}\right\|_2 = 1} \left\| H(j\omega)\begin{pmatrix}\hat{u}\\ \hat{y}\end{pmatrix}\right\|$

we must have $\|H\|_\infty \geq 1$, since $\left\|H(j\omega)\begin{pmatrix}\hat{u}_0\\ \hat{y}_0\end{pmatrix}\right\|_2 = \left\|\begin{pmatrix}\hat{u}_0\\ \hat{y}_0\end{pmatrix}\right\|_2$.
Hence $b_{G,k} \leq 1$.

(e)
$$\|H(j\omega)\|_\infty^2 = \sup_\omega \lambda_{max}\left(\begin{pmatrix}1\\ G\end{pmatrix}(1-kG)^{-1}(1 + kk^*)(1-k^*G^*)^{-1}(1 \;\; G^*)\right)$$
using hint
$$= \sup_\omega \lambda_{max}\left(\left|1 + \frac{2}{j\omega-1}\right|^{-2} 5\left(1 + \frac{1}{j\omega-1}\cdot\frac{1}{-j\omega-1}\right)\right)$$
$$= \sup_\omega 5\left(\frac{|j\omega-1|^2}{|j\omega-1+2|^2}\cdot\frac{(j\omega-1)(-j\omega-1)+1}{|j\omega-1|^2}\right)$$
$$= \sup_\omega 5\,\frac{2+\omega^2}{1+\omega^2} = 10 \Rightarrow b_{G,k} = \frac{1}{\sqrt{10}}$$

3 (a) (i) Advantages: Simple to use. Disadvantages: inconclusive for linearizations that are marginally stable. Only applicable for local stability; cannot be used to prove global asymptotic stability.

(ii) Advantages: Can be used to prove global asymptotic stability and estimate regions of attraction. Disadvantages: it can be difficult to find a Lyapunov function. Lyapunov functions are generally different for different systems.

(b) (i) Setting $\dot{x}_1 = \dot{x}_2 = 0$ we get that the origin is a unique equilibrium point. Linearizing the system about the origin gives

$$A = \begin{bmatrix} -2h(0) & -2h(0) \\ 2 & 0 \end{bmatrix}$$

The poles of the linearized system are the solutions of $\det(sI - A) = 0$. This gives

$$\det(sI - A) = s^2 + 2sh(0) + 4h(0) = 0$$

This has two roots with negative real parts so the origin is asymptotically stable.

(ii)

$$\begin{aligned} V(x_1,x_2) &\geq \int_0^{x_2} z dz + x_1x_2 + x_1^2 \\ &= x_2^2/2 + x_1x_2 + x_1^2 \\ &= \frac{1}{2}[(x_1+x_2)^2 + x_1^2] \geq 0 \end{aligned}$$

(iii)

$$\begin{aligned} \dot{V} &= (2x_1 + x_2)[-2(x_1+x_2)h(x_2)] + (x_2h(x_2) + x_1)2x_1 \\ &= 2[-(2x_1^2 + x_2^2 + 3x_1x_2)h(x_2) + x_1x_2h(x_2) + x_1^2] \\ &= 2[-(x_1+x_2)^2h(x_2) + x_1^2(1 - h(x_2))] \\ &\leq 0 \end{aligned}$$

Also $\dot{V} < 0$ if $(x_1,x_2) \neq (0,0)$ so V is a valid Lyapunov function and asymptotic stability of the origin can be deduced.

(iv) Check if V is radially unbounded, i.e. $(x_1^2 + x_2^2) \to \infty$ implies that $V \to \infty$. This follows from (b)(ii), since the lower bound on V derived, $\frac{1}{2}(x_1+x_2)^2 + x_1^2$, tends to infinity if $|x_1| \to \infty$, or if x_1 is finite and $|x_2| \to \infty$.
Hence the origin is globally asymptotically stable.

version ICI /I

4 (a) Advantages: Easy to check.
Disadvantages: Only applicable to static nonlinearities in feedback interconnection with a linear system. Can be conservative. Cannot be used to estimate the region of attraction.

(b) (i) The linear system has state space realization $\dot{x} = Ax + Bu, y = Cx$ where

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$G(s) = C(sI - A)^{-1}B = \frac{4s+5}{(s+1)^2}$$

(ii) From the circle criterion it is sufficient to verify that $\Re[G(j\omega)] \geq 0$

$$G(j\omega) = \frac{(4j\omega+5)(1-\omega^2-2\omega j)}{(1+\omega^2)^2}$$

So

$$\Re[G(j\omega)] = \frac{8\omega^2+5(1-\omega^2)}{(1+\omega^2)^2} = \frac{3\omega^2+5}{(1+\omega^2)^2} > 0$$

(c) The linear system now has transfer function

$$G(s) = \frac{\lambda}{(s+1)^2}$$

We need to find the smallest value of $\Re[G(j\omega)]$ and ensure that this is greater than -1.

$$G(j\omega) = \frac{\lambda}{(j\omega+1)^2} = \frac{\lambda(1-j\omega)^2}{(1+\omega^2)^2}$$

So

$$\Re[G(j\omega)] = \frac{\lambda(1-\omega^2)}{(1+\omega^2)^2}$$

Differentiating w.r.t. ω^2 and setting this equal to zero we get $\omega^2 = 3$. Hence $\min_\omega \Re[G(j\omega)] = -\lambda/8$. So $0 \leq \lambda < 8$.

END OF PAPER

ASSESSOR'S COMMENTS, MODULE 4F2
Robust and Nonlinear Systems and Control

Q1. Multivariable control. Sensitivity, singular values.

14 attempts, mean 13.4/20 (66.8%), maximum 17, minimum 9.

Candidates were generally able to derive the singular value inequalities and understand their significance. Most knew how to go about analysing the design for the 2-input 2-output compressor problem, but the majority of marks were lost in the last two parts with candidates unable to complete the calculation to find the necessary frequency ranges.

Q2. Gap metric robustness measure.

19 attempts, mean 12.4/20 (61.8%), maximum 19, minimum 6.

A popular question but with rather mixed performance from candidates. The bookwork in part (b) was sometimes poorly done. The simple solutions to (c) and (d) were spotted by some but missed by others. May candidates got stuck part way with the final part (e).

Q3. Lyapunov stability.

21 attempts, mean 12.7/20 (63.5%), maximum 18, minimum 5.

Good answers were generally provided for part (a) (bookwork) and part b(i). Many students had difficulties with part b(ii). There were good attempts for part b(iii) on applying Lyapunov's direct method, though mistakes were often made in the algebra. Very few students provided a correct answer for the last part on global asymptotic stability.

Q4. Circle criterion.

10 attempts, mean 12.2/20 (61.0%), maximum 20, minimum 6.

The question was attempted by half of the candidates. The bookwork and part b(i) were done fairly well, though often with typos in the algebra for part (b)(i). Many students had difficulties in part (c) or did not spend sufficient time to complete it.