4F3: Optimal and Predictive Control Final Crib (2015)

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Version 5 (Post Exam)

1. (a) A function $V(x)$ is a Control Lyapunov equation if: (20%)

   - $V(0) = 0$
   - $V(x) > 0$, for $x \neq 0$
   - There exists a control law $u = \kappa(x)$ such that $V(Ax + Bu) \leq V(x)$.

   Existence of such a function implies that the system $x(k+1) = (Ax(k) + B\kappa(x(k)))$ is stable.

   (b) i. The Value function at time k is defined as: (40%)

   $$V(x(k)) = \min_{u_0,\ldots,u_{N-1}} J(x(k)).$$

   If $x(k)$ is at the origin, the optimal input is $u(k) = 0$. The origin is an equilibrium so $V(0) = 0$.

   The cost function is positive definite. So $V(x) > 0$, $\forall x \neq 0$.

   The corresponding optimal input and state sequences are denoted:

   $$\begin{aligned}\text{Optimal input sequence at time } k\text{:} \quad & (u_0^*, u_1^*, \ldots, u_{N-1}^*)\\ \text{Optimal state sequence at time } k\text{:} \quad & (x_0^*, x_1^*, \ldots, x_{N-1}^*, x_N^*)\end{aligned}$$

   Assume at time k that $u(k) = u_0^*$, and thus $x(k+1) = x_1^*$.

   Consider the *candidate* (non-optimal) predicted sequences at time $k+1$:

   $$\begin{aligned}\text{Candidate input sequence at time } k+1\text{:} \quad & (u_1^*, \ldots, u_{N-1}^*, 0)\\ \text{Candidate state sequence at time } k+1\text{:} \quad & (x_1^*, \ldots, x_{N-1}^*, x_N^*, Ax_N^*).\end{aligned}$$

   The corresponding cost is:

   $$\begin{aligned}J(x_1^*, \tilde{\mathbf{u}}) &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0 - x_N^{*T} P x_N^* + x_N^{*T} Q x_N^* + x_N^{*T} A^T P A x_N^*\\ &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0 + \underbrace{x_N^{*T}(A^T P A - P + Q)x_N^*}_{0}\\ &= V(x_0) - x_0^{*T} Q x_0^* - u_0^{*T} R u_0\\ &\leq V(x_0)\end{aligned}$$

   By optimality: $V(x_1^*) \leq J(x_1^*, \tilde{\mathbf{u}})$. Therefore:

   $$V(x_1^*) \leq V(x_0).$$

   Hence $V(x)$ is a Control Lyapunov Function and this means that the closed loop system is stable.

   ii. If A is open-loop unstable then (15%)

   - Consider a candidate stabilising terminal control law $u_N = Kx_N$.
   - Therefore candidate terminal state for sequence at time $k+1$ is $x_{N+1} = (A + BK)x_N$
   - Therefore need

     $$(A + BK)^T P(A + BK) - P + Q + K^T RK \leq 0.$$

     e.g. sufficient that

     $$(A + BK)^T P(A + BK) - P = -Q - K^T RK.$$

Alternative solution: make P the solution of the discrete-time algebraic Riccati equation and then the (unconstrained) finite horizon MPC with terminal cost is equivalent to the infinite horizon LQR!

(c) We need to show that if $A^T PA - P = -Q$, then

$$x_{N_1}^T P x_{N_1} = \sum_{i=N_1}^{\infty} x_i^T Q x_i.$$

With no terminal control law,

$$\sum_{i=N_1}^{\infty} x_i^T Q x_i = \sum_{i=0}^{\infty} x_{N_1}^T A^{iT} Q A^i x_{N_1} = x_{N_1}^T \left(\sum_{i=0}^{\infty} A^{iT} Q A^i \right) x_{N_1}$$

$$\left(\sum_{i=0}^{\infty} A^{iT} Q A^i \right) = \left(Q + \sum_{i=1}^{\infty} A^{iT} Q A^i \right) \qquad = \left(Q + A^T \left(\sum_{i=0}^{\infty} A^{iT} Q A^i \right) A \right)$$

So,

$$\left(\sum_{i=0}^{\infty} A^{iT} Q A^i \right) = Q + A^T \left(\sum_{i=0}^{\infty} A^{iT} Q A^i \right) A$$

which is consistent with the claim.

This is the discrete-time equivalent of the argument made in the Optimal Control part of the course for continuous-time Lyapunov equations.

An alternative approach was popular with the students. Use the Lyapunov Equation to substitute Q:

$$\begin{aligned} \sum_{i=N_1}^{\infty} x_i^T Q x_i &= \sum_{i=N_1}^{\infty} x_i^T (P - A^T PA) x_i \\ &= \sum_{i=N_1}^{\infty} \left(x_i^T P x_i - x_i A^T PA x_i \right) \end{aligned}$$

Since there is no input applied over this horizon, this is equivalent to:

$$\begin{aligned} \sum_{i=N_1}^{\infty} \left(x_i^T P x_i - x_{i+1}^T P x_{i+1} \right) &= x_{N_1}^T P x_{N_1} \underbrace{- x_{N_1+1}^T P x_{N_1+1} + x_{N_1+1}^T P x_{N_1+1}}_{=0} - x_{N_1+2}^T P x_{N_1+2} + \dots \\ &= \lim_{i \to \infty} x_{N_1}^T P x_{N_1} + x_i^T P x_i \end{aligned}$$

Since A is stable $\lim_{i\to\infty} x_i \to 0$ and thus the condition to be verified has been shown to be true.

2. (a) i. State constraints might represent physical limitations of the system, or operational requirements. For example: (15%)

- Tank level: tank level must not overflow and cannot be lower then completely empty
- Vessel pressure: must not exceed safety limits
- Chemical concentration: Must be within limits
- Temperature: must stay within specific bounds
- Spacecraft position: must maintain visibility constraint with target
- Aircraft airspeed: must remain in safe flight envelope

- Aircraft angle-of-attack: must remain below stall point
- Toy racing car: must stay on track.
- Battery charging current: must not exceed safety limit

ii. - Advantage: modest and tractable online computation (i.e., when the controller operating on the plant) vs. arduous offline computation (i.e., at the design stage) if analytical solution not available (15%)
- Disadvantage: a non-trivial online computation (e.g., solution to QP) must be completed sufficiently fast between sampling instants when the controller is operating on the plant.
- Disadvantage: issues with verification/validation/certification of "correctness" of the software and hardware to implement complex iterative optimisation algorithms in safety critical applications

(b) The difficult part here is the right-hand sides. (35%)

$$\frac{1}{2}(x_i - x_s)^T Q (x_i - x_s) = \frac{1}{2}x_i^T Q x_i + \underbrace{\frac{1}{2}x_s^T Q x_s}_{\text{fixed}} - x_s^T Q x_i$$

$$\frac{1}{2}(u_i - u_s)^T R (u_i - u_s) = \frac{1}{2}u_i^T R u_i + \underbrace{\frac{1}{2}u_s^T R u_s}_{\text{fixed}} - u_s^T R u_i$$

So:

$$H = \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & P \end{bmatrix}, \qquad h = \begin{bmatrix} -Ru_s \\ -Qx_s \\ -Ru_s \\ -Px_s \end{bmatrix}$$

For the equality constraints we have:

$$x_1 = Ax(k) + Bu_0 + d$$
$$x_2 = Ax_1 + Bu_1 + d$$

So:

$$F = \begin{bmatrix} B & -I & 0 & 0 \\ 0 & A & B & -I \end{bmatrix}, \quad f = \begin{bmatrix} -Ax(k) - d \\ -d \end{bmatrix}$$

Finally, for inequality constraints:

$$G = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & M & E & 0 \\ 0 & 0 & 0 & M_N \end{bmatrix}, \quad g = \begin{bmatrix} -Mx(k) + b \\ b \\ b_N \end{bmatrix}$$

Compatible scalar multiples are acceptable as long as they would give the same solution.

(c) The decision vector $\underline{\theta}$ for this QP should be: (35%)

$$\underline{\theta} = \begin{bmatrix} x_s \\ u_s \\ \hat{r} \end{bmatrix}.$$

The equality constraints are

$$\begin{bmatrix} (A - I) & B & 0 \\ C & D & -I \end{bmatrix} \begin{bmatrix} x_s \\ u_s \\ \hat{r} \end{bmatrix} = \begin{bmatrix} -d \\ 0 \end{bmatrix}$$

or equivalent with opposite sign and/or re-ordered rows. The inequality constraints are:

$$\begin{bmatrix} M & E & 0 \\ M_N & 0 & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \\ \hat{r} \end{bmatrix} \leq \begin{bmatrix} b \\ b_N \end{bmatrix}.$$

The cost function is:

$$(\hat{r} - r)^T (\hat{r} - r) = \hat{r}^T \hat{r} + r^T r - 2r^T \hat{r}.$$

So, suitable values for the cost matrices are:

$$H = \begin{bmatrix} 0 & & \\ & 0 & \\ & & I \end{bmatrix}, \qquad h = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}.$$

Alternative solution with a slack variable:

Consider a slack variable z, and let

$$\underline{\theta} = \begin{bmatrix} x_s \\ u_s \\ z \end{bmatrix}$$

The equality constraints are

$$\begin{bmatrix} (A - I) & B & 0 \\ C & D & I \end{bmatrix} \begin{bmatrix} x_s \\ u_s \\ z \end{bmatrix} = \begin{bmatrix} -d \\ r \end{bmatrix}$$

The inequality constraints are identical to the previous case and the cost function is then to minimise $z^T z$, so the cost matrices are

$$H = \begin{bmatrix} 0 & & \\ & 0 & \\ & & I \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Alternative solution condensing out equality constraints:

Assume that there exists a matrix

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} (A - I) & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} d \\ \hat{r} \end{bmatrix}$$

So, we can solve:

$$\min_{\hat{r}} \quad \frac{1}{2} \hat{r}^T \hat{r} - r^T \hat{r}$$

subject to

$$\begin{bmatrix} M\Pi_{12} + E\Pi_{22} \\ M_N \Pi_{12} \end{bmatrix} \hat{r} \leq \begin{bmatrix} b - (M\Pi_{11} + E\Pi_{21})d \\ b_N - (M\Pi_{11})d \end{bmatrix}$$

Then recover x_s, u_s from $\hat{r}$ and d using the above relationship.

3. (a) i. (20%)

$$\|y\|_2 \leq \|G\|_\infty \|w\|_2$$

$$\|y\|_2 = \sqrt{\int_{-\infty}^{\infty} y(t)^T y(t)\,\mathrm{d}t} \qquad \text{Energy in signal}$$

So, the H_∞ norm bounds the maximum gain in the signal energy.

ii. $\|G\|_\infty = \sup_\omega \overline{\sigma}(G(j\omega))$ (10%)

The largest magnitude of the gain of the system over all frequencies.

(b) From the data-sheet, take the Riccati equation: (40%)

$$XA + A^T X + C_1^T C_1 - X(B_2 B_2^T - \gamma^{-2} B_1 B_1^T)X = 0$$

Let:

$$X = \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix}$$

$$A^T X = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x_1 & x_3 \end{bmatrix}$$

$$XA = \begin{bmatrix} 0 & x_1 \\ 0 & x_3 \end{bmatrix}$$

$$C_1^T C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 B_2^T = B_1 B_1^T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So the Riccati equation is:

$$\begin{bmatrix} 0 & x_1 \\ 0 & x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_1 & x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1-\gamma^{-2} \end{bmatrix} \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix}$$

Let $\alpha = 1 - \gamma^{-2}$.

$$\begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix} = \begin{bmatrix} 0 & x_3\alpha \\ 0 & x_2\alpha \end{bmatrix} \begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix} = \begin{bmatrix} x_3^2\alpha & x_2x_3\alpha \\ x_2x_3\alpha & x_2^2\alpha \end{bmatrix}$$

So the Riccati equation is:

$$\begin{bmatrix} 0 & x_1 \\ 0 & x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ x_1 & x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} x_3^2\alpha & x_2x_3\alpha \\ x_2x_3\alpha & x_2^2\alpha \end{bmatrix} = \begin{bmatrix} 1 - x_3^2\alpha & x_1 - x_2x_3\alpha \\ x_1 - x_2x_3\alpha & 2x_3 - x_2^2\alpha \end{bmatrix} = 0$$

Note: some candidates in the exam defined

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^T X = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}, \quad XA = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$$

So

$$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = 0$$

In which case the Riccati equation is

$$\begin{bmatrix} 1 - b^2\alpha & a - bd\alpha \\ a - bd\alpha & 2b - \alpha d^2 \end{bmatrix} = 0$$

Because X is symmetric, this is three equations in three unknowns.

$$1 - x_3^2\alpha = 0 \implies x_3 = \pm\sqrt{\frac{1}{\alpha}} = \pm\alpha^{-1/2}.$$

We want the stabilising solution, so the positive answer is taken.

$$2\alpha^{-1/2} - x_2^2\alpha = 0 \implies x_2^2 = 2\alpha^{-3/2} \implies x_2 = \pm\sqrt{2}\alpha^{-3/4}.$$

Again, take the positive solution because the controller must be stabilising.

Finally (although this is not strictly needed by the question, since it is multiplied by zero to obtain the control law),

$$\begin{aligned} x_1 &= \alpha x_2 x_3 \\ &= \sqrt{2}\alpha\alpha^{-3/4}\alpha^{-1/2} \\ &= \sqrt{2}\alpha^{4/4}\alpha^{-3/4}\alpha^{-2/4} \\ &= \sqrt{2}\alpha^{-1/4}. \end{aligned}$$

So we have:

$$\begin{bmatrix} x_1 & x_3 \\ x_3 & x_2 \end{bmatrix} = \begin{bmatrix} \alpha x_2 x_3 & \alpha^{-1/2} \\ \alpha^{-1/2} & \sqrt{2}\alpha^{-3/4} \end{bmatrix}$$

The optimal control gain (assuming positive feedback) is:

$$K = -B_2^T X = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha x_2 x_3 & \alpha^{-1/2} \\ \alpha^{-1/2} & \sqrt{2}\alpha^{-3/4} \end{bmatrix} = -\begin{bmatrix} \alpha^{-1/2} & \sqrt{2}\alpha^{-3/4} \end{bmatrix} = -\begin{bmatrix} (1-\gamma^{-2})^{-1/2} & \sqrt{2}(1-\gamma^{-2})^{-3/4} \end{bmatrix}$$

So the required control law is:

$$u = -\begin{bmatrix} (1-\gamma^{-2})^{-1/2} & \sqrt{2}(1-\gamma^{-2})^{-3/4} \end{bmatrix} x.$$

(c) i. For example (letting subscripts ub denote, "upper bound" and lb denote "lower bound": (15%)

- Start with $\gamma = 1$.
- If infeasible
  - keep doubling γ until feasible. Call this value γ_{ub} and let $\gamma_{lb} = 1$.
- Else
  - Keep halving γ until infeasible. This is γ_{lb}. Let $\gamma_{ub} = 1$.
- $\gamma^* \in [\gamma_{lb}, \gamma_{ub}]$
- Use interval bisection to find minimum feasible value of γ^*

Alternatively suggest a direct line search at a pre-determined degree of granularity (potentially less efficient)

ii. The data-sheet has formulae for extending to the output-feedback case. However, there are assumptions with respect to controllability and observability. In this case, the pair (A, C_2) is not observable. (15%)

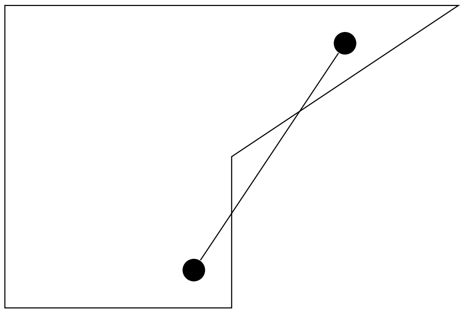
The observability matrix is:

$$\begin{bmatrix} C_2 \\ C_2 A \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \implies \text{not full rank so not observable.}$$

Because the system is not observable, designing the controller for the new value of C_2 is not appropriate.

4. (a) i. A function $f(x)$ is convex if $f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y)$, for $\alpha \in [0,1]$. (10%)

ii. (10%)



There exist two points inside the set which cannot be linked by a single straight line segment that stays wholly inside the set.

iii. Convexity is useful because a local optimum is also the global optimum. Moreover, efficient algorithms can be used to find the optimal solution numerically. (10%)

(b) i. We cannot assume $S = 0$, so we need: (10%)

$$X_2 \geq 0$$
$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0.$$

Suggested division of marks: 5% for each point.

ii. (20%)

$$B^T X_2 A = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} 100 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 50 & 20 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 50 & 70 \end{bmatrix}.$$

$$R + B^T X_2 B = 5 + \begin{bmatrix} 50 & 20 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = 5 + 45 = 50.$$

$$u_1 = -(R + B^T X_2 B)^{-1} B^T X_2 B A x_1 = -\frac{1}{50} \begin{bmatrix} 50 & 70 \end{bmatrix} x_1 = -\begin{bmatrix} 1 & 7/5 \end{bmatrix} x_1$$

iii. For $k = 2$: (20%)

$$V(x, 2) = x^T X_2 x, \quad X_2 = \begin{bmatrix} 100 & 0 \\ 0 & 20 \end{bmatrix}.$$

For $k = 1$ we need to do an iteration of the Riccati difference equation (found in the data sheet):

$$X_{k-1} = Q + A^T X_k A - A^T X_k B (R + B^T X_k B)^{-1} B^T X_k A.$$

We've already computed part of this in the previous subpart.

$$X_1 = Q + A^T X_2 A + A^T X_2 B K_1$$

$$A^T X_2 A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 100 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 100 & 100 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} 100 & 100 \\ 100 & 120 \end{bmatrix}.$$

From above,

$$A^T X_2 B = (B^T X_2 A)^T = \begin{bmatrix} 50 \\ 70 \end{bmatrix}$$

$$\begin{aligned}
X_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 100 & 100 \\ 100 & 120 \end{bmatrix} - \begin{bmatrix} 50 \\ 70 \end{bmatrix} \begin{bmatrix} 1 & 7/5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 100 & 100 \\ 100 & 120 \end{bmatrix} - \begin{bmatrix} 50 & 70 \\ 70 & 98 \end{bmatrix} \\
&= \begin{bmatrix} 51 & 30 \\ 30 & 23 \end{bmatrix}.
\end{aligned}$$

and $V(x, 1) = x^T X_1 x$. If $x_1 = [1, 1]^T$, then $V(x_1, 1) = 134$.

(c) The discrete-time finite horizon linear quadratic regulator computes a time varying control law over a finite horizon. (20%)

The predictive controller makes a prediction using the same cost function. The *predicted trajectory* will coincide with the trajectory obtained using the finite horizon LQR starting from the measured state. However, only the first element of the control sequence will be applied to the plant.

- At time $k = 0$, both controllers will apply the same control action.
- At time $k = 1$, however, the finite horizon LQR will apply $u_1 = K_1 x_1$ whereas the predictive controller employs a "receding" horizon, and a new prediction is made from x_1 and thus, the MPC will apply $u_1 = K_0 x_1$.

Comments from Assessor's Report

Q1 Stability of Unconstrained MPC

17 attempts, average mark 10.41/20, Maximum 19, Minimum 0.

A relatively popular question, but with some disappointing answers. The final part (c), which drew together material from two parts of the course was well answered by most candidates, but answers to part (b) were disappointing despite this being essentially routine "book work". Some candidates failed to notice that part (a) concerned "Control Lyapunov Functions" and not simply "Lyapunov Functions", although the penalty for this was only 1 mark.

Q2 QP for Uncondensed Constrained MPC and Constrained target Calculation

22 attempts, average mark 12.14/20, Maximum 18, Minimum 8.

The jointly most popular question. Parts (a) and (b) were generally well answered. Some candidates mentioned actuator positions when asked to provide examples of "state constraints". These were given the benefit of the doubt on the basis that the actuator slew rate is often the modelled input of the system in MPC applications. Conversely, part (c) on constrained target calculation was found more challenging, with only a handful of good answers, and many candidates ignored the inequality constraints, despite these being the primary motivation for the question.

Q3 State Feedback $\mathcal{H}_\infty$ Control

8 attempts, average mark 13.13/20, Maximum 20, Minimum 8.

The least popular question, but with a relatively high average mark. This question required recall of the definition of the $\mathcal{H}_\infty$, application of a Riccati equation from the data sheet to design a state-feedback control law, and to show that a modified system was not observable. Some candidates did not notice that part (b) did not require an observer design, since full state feedback was available, and would have lost time as a result.

Q4 Convexity and Discrete-Time LQR

22 attempts, average mark 11.09/20, Maximum 20, Minimum 3.

The jointly most popular question. Part (a) on convex functions and sets was well answered. Part b(ii) was well answered, but answers to Part b(iii) were prone to numerical slips. Some candidates failed to notice that the Riccati equation from the datasheet was applicable. Most candidates were able to explain the difference between finite horizon DLQR and unconstrained receding horizon MPC, although (worryingly) some were of the opinion that the unconstrained MPC might have active constraints.