EGT4
ENGINEERING TRIPOS PART IIB

Monday 18 April 2016   2 to 3.30

Module 4F3

OPTIMAL AND PREDICTIVE CONTROL — SOLUTIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Consider the active shock absorber in Figure 1. The dynamics of the system satisfies the balance of forces $ma + kp + cv = f_h$, where a is the acceleration of the mass m, p is the displacement of the mass, v is the velocity of the mass, f_h is the force exerted by a hydraulic actuator, k is the spring constant, and c is the damping coefficient. The actuator dynamics are approximated by a first order lag $\tau\left(\frac{d}{dt}f_h\right) = -f_h + u$ with time constant $\tau > 0$ and input u.

(a) (i) Given the state vector $x = \begin{bmatrix} p & v & f_h \end{bmatrix}^T$ and measured output $y = p$, show that the overall system behaviour is characterised by the linear equations $\dot{x} = Ax + Bu$, $y = Cx + Du$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & \frac{1}{m} \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 0\,. \quad (1)$$

[5%]

(ii) Show that this system is stable. [10%]

(b) Let Q be the infinite-time observability gramian: $Q = \int_0^\infty e^{A^T t} C^T C e^{At}\, dt$.

(i) Show that Q is a solution to the Lyapunov equation $A^T Q + QA + C^T C = 0$. [20%]

(ii) If $Q > 0$, show that (A, C) is observable (without computing the observability matrix). [20%]

(c) (i) What is meant by a *balanced realisation* of a transfer function, and by its *Hankel singular values*? [15%]

(ii) Explain how a balanced realisation of a stable system can be used to obtain lower-order approximations of the system. [15%]

(iii) Let $G(s) = C(sI - A)^{-1}B$, where A, B, C are defined in (1) for some particular parameter values. $\hat{G}(s)$ is a second-order approximate model of $G(s)$, obtained from a balanced realisation. Figure 2 shows the Bode magnitude plot of $E(s) = G(s) - \hat{G}(s)$. Estimate $\|E\|_\infty$, and hence obtain lower and upper bounds for the third Hankel singular value of $G(s)$. [15%]

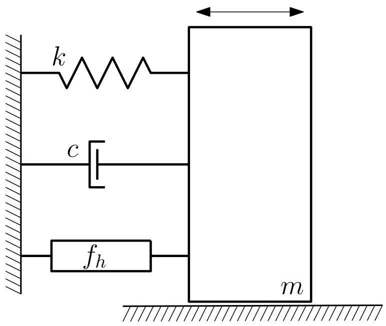

Fig. 1

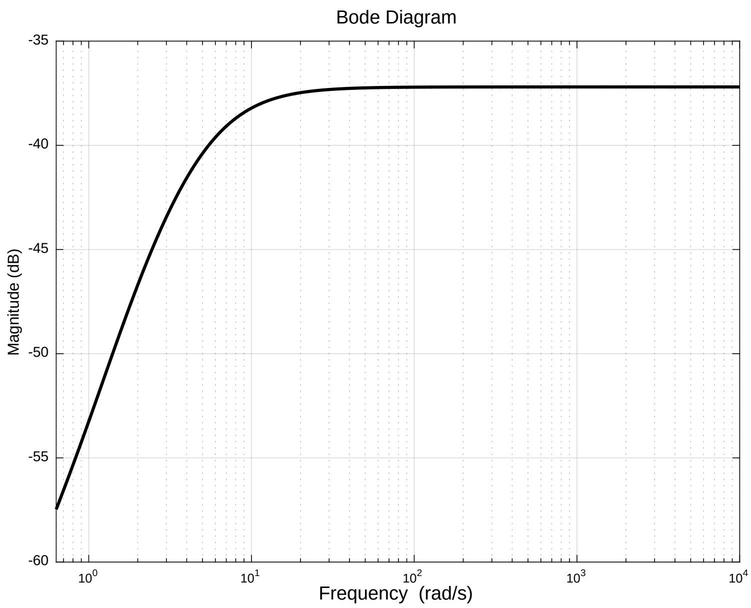

Fig. 2

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SOLUTION:

(a) (i) Use the fact that $a = \dot{v}$ and $v = \dot{p}$. Then we immediately have the first row of A and B. The second row comes from $\dot{v} = -kp/m - cv/m + f_h/m$, and the third row comes from the actuator lag: $\dot{f}_h = -f_h/\tau + u/\tau$. The C and D matrices come directly from the choice $y = p$.

(ii) To establish stability, find the eigenvalues of A. Expanding the determinant by the third row gives $\det(sI - A) = (s + 1/\tau)[s(s + c/m) + k/m]$. So one eigenvalue is at $-1/\tau < 0$ (since the time constant must be positive) and the other two are at the roots of $s^2 + (c/m)s + k/m$, which have negative real parts since all coefficients of the polynomial have the same sign (special case of the Routh-Hurwitz criterion for quadratic polynomials), or by direct calculation using the formula for roots of a quadratic polynomial. (We have used the fact that c, k and m are all positive.)

(b) (i) Show that $Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ is a solution to the Lyapunov equation $A^T Q + QA + C^T C = 0$ by substitution:

$$
\begin{aligned}
A^T Q + QA + C^T C &= \int_0^\infty A^T e^{A^T t} C^T C e^{At} + e^{A^T t} C^T C e^{At} A dt + C^T C \\
&= \left[e^{A^T t} C^T C e^{At} \right]_0^\infty + C^T C = -C^T C + C^T C = 0 \, .
\end{aligned}
$$

Note that $e^{At} \to 0$ *as* $t \to \infty$ *since we already know that the system is stable.*

(ii) Suppose that the pair (A, C) is not observable. Then there exists a state $x \neq 0$ such that $Ce^{At}x = 0$ for all $t \geq 0$. Thus,

$$
\begin{aligned}
0 &= \int_0^\infty x^T e^{A^T t} C^T C e^{At} x dt \\
&= x^T \int_0^\infty e^{A^T t} C^T C e^{At} dt x \\
&= x^T Q x \qquad \text{for } x \neq 0
\end{aligned}
$$

which contradicts $Q > 0$.

(c) (i) A *balanced realisation* of a transfer function is one for which the observability gramians and the controllability gramians are equal and diagonal, namely

$$
P = Q = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \sigma_n \end{bmatrix}
$$

where Q is as in part (b), $P = \int_0^\infty e^{At} BB^T e^{A^T t} dt$ is the controllability gramian, and it is conventional to order the diagonal entries so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. ($n$ is the dimension of the state space.) The (nonnegative) real numbers $\sigma_1, \ldots, \sigma_n$ are called the *Hankel singular values* of the system.

(ii) If the Hankel singular values are ordered as in part (i), then a lower-order approximation of the system can be obtained by truncating the state vector, retaining only the initial part and discarding the rest. The corresponding parts of the matrices A, B, C are retained or discarded. It can be shown that if the original system is stable, then the 'reduced' (ie approximate) system is also stable.
(Justification — not asked for in the question: state variables corresponding to small Hankel singular values are not much affected by the input, and have little effect on the output. Hence they play little role in the input- output behaviour, so can be omitted without changing the input-output behaviour much. In the limit, if a Hankel singular value were zero, the corresponding state variable would be uncontrollable and unobservable, and so could be omitted without changing the input-output behaviour at all.)

(iii) Since $G(s)$ is stable (from part (a)(ii)), $\hat{G}(s)$ and hence $E(s)$ are stable too. Therefore $\|E\|_\infty = \sup_\omega |E(j\omega)|$ (noting that $E(s)$ is a scalar transfer function, since the system $G(s)$ has only one input and output). Hence from Figure 4 we get $\|E\|_\infty \approx -37\text{dB} = 10^{-37/20} = 0.014$.
In general, if the approximate system is obtained by discarding states $k+1, \ldots, n$ in the balanced realisation, the standard result is that

$$\sigma_{k+1} \leq \|E\|_\infty \leq 2(\sigma_{k+1} + \sigma_{k+2} + \cdots + \sigma_n).$$

In this case we have $k = 2$ and $n = 3$, so we have $\sigma_3 \leq 0.014 \leq 2\sigma_3$. Hence $0.007 \leq \sigma_3 \leq 0.014$, approximately.
Note: If the estimation is done more accurately then $\|E\|_\infty = -37.5\text{dB} = 0.0133$, which gives $0.0067 \leq \sigma_3 \leq 0.0133$. The actual value of σ_3 is 0.0069.

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2 A mass-spring-damper system with a unit mass is described by the state-space equations

$$\dot{x} = Ax + Bu + Bw = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \quad y = Cx = \begin{bmatrix} 1 & 0 \end{bmatrix} x \qquad (4)$$

where $u = Kx = -[k, c]x$ is the force acting on the mass, y is the position of the mass, k is the spring stiffness and c is the damper coefficient.

(a) Take $w = 0$ and consider the cost

$$\int_0^\infty [x(t)^T Qx(t) + u(t)^T Ru(t)]dt, \quad \text{where} \quad Q = C^T C \text{ and } R = 1\ . \qquad (5)$$

(i) Show that the values of k and c which minimise the cost (5) are given by $\begin{bmatrix} k & c \end{bmatrix} = B^T X$ for $X = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$. [25%]

(ii) Noting that (A, B) is controllable and (A, C) is observable, what conclusion can be drawn about the stability of $(A + BK)$, if $K = -[k, c]$ is chosen as in part (i)? [10%]

(b) Let u and y be the input and output signal vectors of a system with transfer function $G(s)$.

(i) The infinity norm $\|G\|_\infty$ satisfies the relationship $\|G\|_\infty = \sup \|y\|_2 / \|u\|_2$ where the supremum is taken over all non-zero inputs with $\|u\|_2 < \infty$. How are the signal norms $\|u\|_2$ and $\|y\|_2$ defined? [10%]

(ii) If $V(x) = x^T X x$ for some $X = X^T > 0$, $G(s) = C(sI - A)^{-1}B$, and X satisfies the Riccati equation

$$A^T X + XA + C^T C + \frac{1}{\gamma^2} XBB^T X = 0$$

then it can be shown that

$$\frac{dV}{dt} + y^T y - \gamma^2 u^T u \leq 0. \qquad (6)$$

Show that, if (6) holds, then $\|G\|_\infty \leq \gamma$. [20%]

(iii) For the system defined in (4), let $H(s)$ be the transfer function from w to y. If the coefficients k and c are chosen as in part (a)(i), show that $\|H\|_\infty \leq 1$. [35%]

SOLUTION:

(a) (i) This is the infinite-horizon LQR problem. From the Data Sheet we know that for the finite-time LQR problem the optimal solution is $u(t) = K(t)x(t)$ where $K(t) = -R^{-1}B^TX(t)$ and $X(t)$ solves the RIccati differential equation $-\dot{X} = Q + XA + A^TX - XBR^{-1}B^TX$. But we know that $X(t)$ becomes constant in the limit of the horizon becoming infinite, so for the given problem we need to solve the algebraic Riccati equation (ARE) obtained by setting $\dot{X} = 0$:

$$0 = Q + XA + A^TX - XBR^{-1}B^TX$$

and apply the constant-gain state feedback $u(t) = Kx(t)$ where $K = -R^{-1}B^TX$. We can check that the X given in the question satisfies the ARE, noting that $Q = C^TC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 1$:

$$\begin{aligned}
&Q + XA + A^TX - XBB^TX \;= \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} - \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}^2 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 1 \end{bmatrix} - \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1-1 & \sqrt{2}-\sqrt{2} \\ \sqrt{2}-\sqrt{2} & 1+1-2 \end{bmatrix} \;=\; 0
\end{aligned}$$

We should also check that $X > 0$. This holds, because $X_{1,1} = \sqrt{2} > 0$ and $\det X = 2 - 1 > 0$.
Note: The actual values of k and c (not asked for in the question) are given by:

$$-K = \begin{bmatrix} k & c \end{bmatrix} = B^TX = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}$$

In the exam several candidates solved the ARE as if they didn't already know the answer. It is harder and longer to do it that way.

(ii) From infinite-horizon LQR theory we know that $A + BK$ is guaranteed to be stable if (A, B) is controllable and (A, C) is observable. *Note:* That is a sufficiently good answer; a better answer is that the infinite- horizon cost would be infinite if $A + BK$ were not stable.

(b) (i) $\|u\|_2 = \sqrt{\int_0^\infty u(t)^Tu(t)dt}$ and $\|y\|_2$ is defined similarly.

(ii) If $\dot{V} + y^T y - \gamma^2 u^T u \leq 0$ then, integrating,

$$V(x(\infty)) - V(x(0)) + \int_0^\infty y(t)^T y(t) dt - \gamma^2 \int_0^\infty u(t)^T u(t) dt = \\ V(x(\infty)) - V(x(0)) + \|y\|_2^2 - \gamma^2 \|u\|_2^2 \leq 0. \quad (8)$$

But A must be stable since $X > 0$ and $C^T C + \frac{1}{\gamma^2} X B B^T X \geq 0$ (think of the Riccati equation as a Lyapunov equation) and we assume $\|y\|_2 < \infty$, so $V(x(\infty)) = 0$. Also for calculation of input-output norms we must assume zero initial conditions (otherwise arbitrarily large contributions to $\|y\|_2$ could arise from $x(0)$), so we should take $V(0) = 0$. Hence $\|y\|_2^2 - \gamma^2 \|u\|_2^2 \leq 0$ for all u, so $\|y\|_2 / \|u\|_2 \leq \gamma$, and hence (by definition) $\|G\|_\infty \leq \gamma$.

(iii) Since $u = Kx$ we have $\dot{x} = (A + BK)x + Bw$ and $y = Cx$, so $H(s) = C(sI - A - BK)^{-1} B$. So, using the Riccati equation given in part (b)(ii) of the question with $\gamma = 1$, we can show that $\|H\|_\infty \leq 1$ if we can show that there exists a $Y = Y^T > 0$ such that

$$(A + BK)^T Y + Y(A + BK) + C^T C + Y B B^T Y = 0.$$

Also we know that $K = -B^T X$, where X solves the LQR Riccati equation (as in part (a)(i)). So we have (noting that X and Y are symmetric)

$$(A + BK)^T Y + Y(A + BK) + C^T C + Y B B^T Y = \\ (A - B B^T X)^T Y + Y(A - B B^T X) + C^T C + Y B B^T Y = \\ A^T Y + Y A - X B B^T Y - Y B B^T X + C^T C + Y B B^T Y \quad (9)$$

Now trying the solution $Y = X$ gives

$$A^T X + X A - X B B^T X - X B B^T X + C^T C + X B B^T X = \\ A^T X + X A + C^T C - X B B^T X = 0 \quad (10)$$

from part (a)(i). Thus the condition given in part (b)(ii) holds with $\gamma = 1$, and so $\|H\|_\infty \leq 1$.

Many (most?) candidates in the exam assumed that X and Y are the same. But two different Riccati equations are needed in parts (a) and (b), and it is not obvious that the same solution satisfies both.

3 (a) (i) How is a *convex set* defined? [15%]

(ii) When is an optimisation problem convex? [15%]

(iii) Give two examples of convex optimisation problems. [15%]

(b) A plant with state x_k and input u_k at time k is described by the discrete-time state-space model $x_{k+1} = Ax_k + Bu_k$. A predictive controller minimises the cost function

$$V(x_0, u_0, u_1, \ldots, u_{N-1}) = x_N^T P x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k)$$

subject to the constraints $Mx_k + Eu_k \leq b$ for $k = 0, 1, \ldots, N-1$. Show that this problem can be written as a standard quadratic programming problem of the form

$$\text{minimise } \theta^T H \theta \quad \text{subject to} \quad F\theta - f = 0 \quad \text{and} \quad G\theta - g \leq 0$$

for suitable matrices F, G, H and suitable vectors f, g, with the vector θ containing the decision variables $u_0, u_1, \ldots, u_{N-1}$ and $x_1, x_2, \ldots, x_N$.

(c) A 'terminal constraint' of the form $M_N x_N \leq b_N$ is sometimes added to predictive control problems. Comment briefly (without technical details) on the reason for adding such a constraint, and on the properties that it should satisfy.

SOLUTION:

(a) (i) If S is a set in Euclidean space, and x_1, x_2 are any two points in S, then S is convex if and only if $\alpha x_1 + (1-\alpha)x_2$ is in S, where $0 \leq \alpha \leq 1$.
Very few candidates knew this way of expressing convexity of a set, but most expressed it correctly in words ('a straight line joining any two points ... '). Some defined convexity of a function instead of a set.

(ii) An optimisation problem is convex when both the objective function and the feasible set are convex. *Note:* There are various ways of expressing this, eg: if the problem is $\min_x f(x)$ subject to $g(x) \leq 0$ then the problem is convex if both $f(.)$ and $g(.)$ are convex functions.

(iii) Two examples of convex optimisation problems are (1) linear programming problems and (2) convex quadratic programming problems. *Note:* Mathematical formulations are equally good answers, eg: (1) $\min_x c^T x$ subject to $Ax \leq b$, (2) $\min_x x^T Hx + c^T x$ subject to $Ax \leq b$ with $H \geq 0$. (The condition on H is important!)
This question was unintentionally ambiguous. Some candidates gave examples of predictive control applications ('flying an aircraft straight and level', 'controlling

the quality and strength of paper' etc). Reasonable answers along these lines were accepted.

(b) Let $\theta^T = [u_0^T, x_1^T, u_1^T, x_2^T, \ldots, x_{N-1}^T, u_{N-1}^T, x_N^T]$ (that is, include $x_1, x_2, \ldots$ as decision variables, even though they are not needed in the solution). Note that x_0 is *not* a decision variable. Then the cost function V is quadratic in the elements of θ, so it can be represented by the term $\theta^T H \theta$ *(strictly speaking, it would be enough to stop the explanation there)*, where

$$H = \begin{bmatrix} R & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & R & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & Q & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & Q & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & R & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & P \end{bmatrix}$$

Equality constraints arise from $x_{k+1} = Ax_k + Bu_k$ which can be written as $Ax_k + Bu_k - x_{k+1} = 0$ for $k = 0, 1, \ldots N$, or $F\theta - f = 0$, where $f^T = [(Ax_0)^T, 0, 0, \ldots, 0]$ and

$$F = \begin{bmatrix} B & -I & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & A & B & -I & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & A & B & -I \end{bmatrix}$$

The inequality constraints $Mx_k + Eu_k \leq b$ for $k = 0, 1, \ldots, N-1$ can be written as $G\theta - g \leq 0$, where $g^T = [(Mx_0 - b)^T, -b^T, -b^T, \ldots, -b^T, 0]$ and

$$G = \begin{bmatrix} E & 0 & 0 & 0\ldots & 0 & 0 & 0 & \\ 0 & M & E & 0\ldots & 0 & 0 & 0 & \\ 0 & 0 & 0 & M\ldots & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & M & E & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

Note that there is no constraint on x_N.

(c) A terminal constraint is sometimes added to ensure recursive feasibility of a predictive controller, ie to ensure that once a feasible solution has been found, the problem remains feasible at all future time steps. To achieve this, the terminal constraint should be constraint admissible (ie should be achievable with inputs and states which are within constraints) and should be positively-invariant under the action of the terminal control law implied by the model and the cost function.

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4 An unconstrained predictive controller determines the input signal $u(k)$ at time k by minimising the cost function

$$V(x(k),\mathbf{u}) = x_N^T P x_N + \sum_{i=0}^{N-1} (x_i^T Q x_i + u_i^T R u_i)$$

where $x(k)$ is the measured current state, $x_0 = x(k)$,

$$x_{i+1} = Ax_i + Bu_i \quad \text{for} \quad i = 0, 1, \ldots, N-1$$

and setting $u(k) = u_0^*$, where the minimising input sequence is

$$\mathbf{u}^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*).$$

(a) Explain why repeated determination of the input signal in this manner results in a feedback system. [10%]

(b) What is meant by the phrase *terminal cost* in this context? [10%]

(c) Assume that $P > 0$, $Q > 0$ and $R > 0$, and that K is a matrix such that all the eigenvalues of $A+BK$ lie within the unit circle. Show that, if

$$(A+BK)^T P (A+BK) - P \leq -Q - K^T R K$$

then the origin ($x = 0$) of the closed-loop system with the predictive controller is asymptotically stable. [50%]

(d) A particular 1-input, 1-state system has $A = 1.2$, $B = 1$, and the control system designer chooses $Q = 5$ and $R = 2$. Find a P which results in an asymptotically stable closed-loop system. [30%]

SOLUTION:

(a) The next state $x(k+1)$ will be mostly determined by the input $u(k) = u_0^*$ (not completely, because of disturbances and model errors). The next input $u(k+1)$ will be computed on the basis of the measured value of $x(k+1)$. So the input depends on the latest measurement, which defines a feedback system.

(b) *Terminal cost* refers to the term $x_N^T P x_N$ in the cost function of the predictive controller.

(c) The strategy is to show that if the inequality is satisfied then the value function $V^*(x(k)) = \min_{\mathbf{u}} V(x(k), \mathbf{u})$ is a Lyapunov function for the closed-loop system. We assume that the model represents the system behaviour exactly.
At time k we have the optimal solution $\mathbf{u}^* = (u_0^*, u_1^*, \ldots, u_{N-1}^*)$. At the next step we could apply the input sequence

$$\tilde{\mathbf{u}} = (u_1^*, u_2^*, \ldots, u_{N-1}^*, \tilde{u}_N)$$

for some input $\tilde{u}_N$ to be applied at the end of the horizon. Furthermore, we could choose $\tilde{u}_N = Kx_N$, which would give a terminal state $x(k+N+1) = Ax_N + BKx_N = (A+BK)x_N$. The cost obtained with this choice of input is

$$V(x(k+1), \tilde{\mathbf{u}}) = x(k+N+1)^T P x(k+N+1) + \sum_{i=1}^{N-1} (x_i^{*T} Q x_i^* + u_i^{*T} R u_i^*) + x_N^{*T} Q x_N^* + x_N^{*T} K^T R K x_N^* \quad (11)$$

where x_i^* denotes the state at step $k+i$ resulting from applying the input sequence $\mathbf{u}^*$ at time k. This differs from $V^*(x(k))$ by subtracting terms corresponding to the old first step and adding terms corresponding to the new last step:

$$\begin{aligned}
V(x(k+1), \tilde{\mathbf{u}}) &= V^*(x(k)) - x_N^{*T} P x_N^* - x_0^T Q x_0 - u_0^{*T} R u_0^* \\
&\quad + x(k+N+1)^T P x(k+N+1) + x_N^{*T} Q x_N^* + x_N^{*T} K^T R K x_N^* \\
&= V^*(x(k)) - x_N^{*T} P x_N^* - x_0^T Q x_0 - u_0^{*T} R u_0^* \\
&\quad + x_N^{*T} (A+BK)^T P (A+BK) x_N^* + x_N^{*T} (Q + K^T R K) x_N^* \\
&< V^*(x(k))
\end{aligned}$$

if

$$x_N^{*T} [-P + (A+BK)^T P (A+BK) + Q + K^T R K] x_N^* \leq 0$$

(since $Q > 0$ and $R > 0$) which will be true if

$$-P + (A+BK)^T P (A+BK) \leq -Q - K^T R K.$$

 (cont.

The requirement that K should be stabilising comes from the fact that $P > 0$ can only satisfy this inequality if $A + BK$ has eigenvalues within the unit circle, since $Q > 0$ and $R > 0$ (from linear systems stability theory).

Now the input $u(k+1)$ will be chosen by optimising the cost function (rather than by using the input $\tilde{\mathbf{u}}$), which will give the value function

$$V^*(x(k+1)) \leq V(x(k+1), \tilde{\mathbf{u}}) < V^*(x(k))$$

if the required inequality holds. But $V^*(x(k)) \geq 0$, and $V^*(x(k)) = 0$ only if $x(k) = 0$, since $P > 0, Q > 0, R > 0$. Hence $V^*(x(k))$ is a (discrete-time) Lyapunov function for the closed-loop system, and closed-loop asymptotic stability of the origin is established.

The solutions to this part resembled the old joke about 'playing all the right notes, but not in the right order'. Correct bits of algebra appeared in most solutions, and the phrase 'Lyapunov function' was also present in most, but very few candidates followed the logic of the proof correctly.

(d) Since $A = 1.2$ and $B = 1$, K is a scalar and $|A+BK| < 1 \Leftrightarrow |1.2+K| < 1 \Leftrightarrow -2.2 < K < -0.2$. We can choose any K in this range, and then determine an allowable range for values of P.

For example, let $K = -1.2$, then $A + BK = 0$ so the inequality in part (c) becomes $-P < -5 - 2 \times (1.2)^2 = -7.88$ so any $P > 7.88$ will do, for example $\underline{P = 8}$.

For other values of K, the inequality remains linear in P, eg: $K = -1 \Leftrightarrow A + BK = 0.2$ so we need $(0.2)^2 P - P < -5 - 2 = -7$, hence $P > 7/0.96 = 7.29$.

Notes for marking part (d): Candidates can select any value of K in the stabilising range, then determine the allowed range of P. To help with checking the solutions, Figs.3 and 4 show the minimum allowed P value for each K. Also Table (d) gives the minimum for a range of K values. Some credit will be given for arguing that a large enough P will always satisfy the inequality for some K (since $|A+BK| < 1$), but no credit will be given for simply guessing a very large value of P without checking it — since there is a minimum possible value (approx. 7.3), as can be seen from Fig.4.

K	-2.1	-2.0	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4	-1.3	-1.2
P	72.74	36.11	23.96	17.94	14.37	12.05	10.44	9.29	8.46	7.88
K	-1.1	-1.0	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	
P	7.49	7.29	7.27	7.48	7.97	8.94	10.78	14.78	27.26	

Table 1

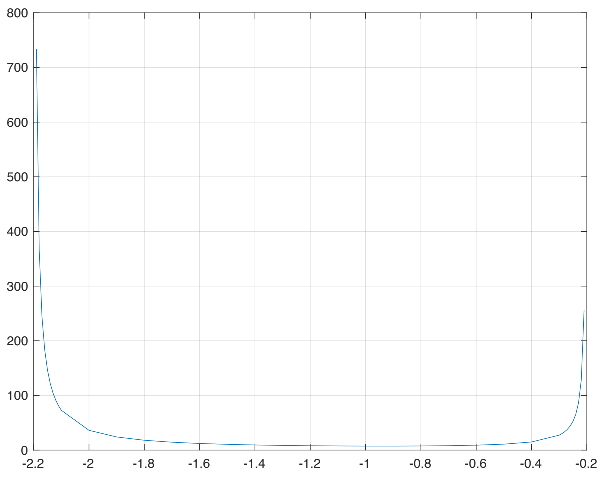

Fig. 3

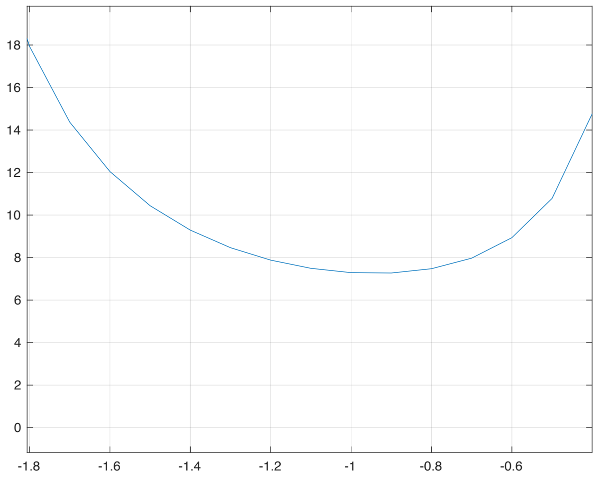

Fig. 4

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