DR SUMEETPAL SINGH PART IIB 2015 ①

4F7 DIGITAL FILTERS AND SPECTRUM ESTIMATION

Question 1

a)
$$(x(n) - h^T \underline{u}(n))^2$$
$$= x(n)^2 + \underline{h}^T \underline{u}(n) \underline{u}(n)^T \underline{h} - 2 \underline{h}^T \underline{u}(n)\, x(n)$$

$$E\{LHS\} = 1 + \underline{h}^T \underline{R}\, \underline{h} - 2 \underline{h}^T \underline{p}$$

$$\nabla_{\underline{h}} = 0 \quad \text{gives}$$

$$2 R \underline{h} = 2 \underline{p}$$
$$\underline{h} = \underline{R}^{-1} \underline{p}$$

b) (i)
$$E\{u(n) x(n)\} = a_0 E\{x(n)\}^2 + \sum_{m=1}^{M-1} a_m E\{x(n-m) x(n)\} + E\{x(n) w(n)\}$$
$$= a_0$$

For $i > 0$, $E\{u(n-i) x(n)\} = 0$

because $u(n-i)$ has no $x(n)$ term within. Thus

$$\underline{p} = \begin{bmatrix} a_0 \\ 0 \\ \vdots \end{bmatrix}$$

(ii)
$$z(n) = \sum_{m=0}^{M-1} a_m x(n-m)$$

$$E\{z(n)^2\} = \frac{1}{M^2} \sum_{m=0}^{M-1} E\{x(n-m)\}^2 + \text{cross terms}$$

$$= \frac{1}{M}$$

We see $z(n)$ is a MA process

$$E\{z(n+k)z(n)\} = \begin{cases} \dfrac{M-|k|}{M^2} & |k|<M \\ 0 & |k| \geq M \end{cases}$$

(iii) $$\underline{R} = E\left\{ \begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(n-L+1) \end{bmatrix} \begin{bmatrix} u(n) & u(n-1) & \cdots & u(n-L+1) \end{bmatrix} \right\}$$

Let $r_u(k) = E\{u(n)u(n+k)\}$

row 1 of $\underline{R}$ is $[r_u(0) \ r_u(1) \ \ldots \ r_u(L-1)]$

row 2 is $[r_u(1) \ r_u(0) \ \ldots \ r_u(L-2)]$

similarly for the rest

$$r_u(k) = \underbrace{r_z(k)}_{E\{z(n+k)z(n)\}} + \underbrace{r_w(k)}_{E\{w(n+k)w(n)\}}$$

since $u(n) = z(n) + w(n)$

Thus row 1 of $\underline{R}$ is $\left[\sigma^2 + \dfrac{M}{M^2}, \dfrac{M-1}{M^2}, \dfrac{M-2}{M^2}, \cdots \dfrac{1}{M^2}, 0 \cdots 0\right]$

assuming $L > M$

(c) LMS implementation is stochastic gradient descent

$$\underline{h}(n+1) = \underline{h}(n) + \mu\, \underline{u}(n)\, e(n)$$

$$e(n) = x(n) - \underline{h}^T(n)\, u(n)$$

$$\mu = \text{step-size}$$

(d) if a_m were time varrying then

a larger step-size μ will track the changes in the channel.

Large μ gives quick tracking but noisier $\underline{h}_{opt}(n)$ estimates - oscillations of $\underline{h}(n)$ about $\underline{h}_{opt}(n)$ are large.

Smaller μ may result in $\underline{h}(n)$ not tracking $\underline{h}_{opt}(n)$ quick enough

Care must be taken to ensure LMS does not diverge because too large a step-size was used. Remedy would be the NLMS.

(e) $0 < \mu < \dfrac{2}{\lambda_{max}(R)} < \dfrac{2}{\text{trace}(R)}$

$$\text{trace}(R) = \left(\sigma^2 + \frac{1}{M}\right) \times L$$

if : $a_m = \dfrac{1}{m}$.

For a general a_m

$$\text{trace}(R) = \left(\sigma^2 + \sum_{m=0}^{M-1} a_m^2\right) \times L.$$

Question 2

(a) $E\left\{\left(y(n) - \underline{h}^T \underline{u}(n) \right)^2 \right\}$

Minimiser is $R^{-1} p$

where $R = E\{ \underline{u}(n) \underline{u}(n)^T \}$

$p = E\{ y(n) \underline{u}(n) \}$

You must derive these — see lecture notes

$\underline{u}(n) = \begin{bmatrix} u(n) \\ u(n-1) \end{bmatrix} \Rightarrow R = \sigma^2 \underline{I}$

$y(n) u(n) = x(n) u(n) + z(n) u(n)$

$E\{ y(n) u(n) \} = 2\sigma^2$

$y(n) u(n-1) = x(n) u(n-1) + z(n) u(n-1)$

$x(n) u(n-1) = u(n) u(n-1) + \alpha\, x(n-1)\, u(n-1)$

$E\{ x(n) u(n-1) \} = \alpha \sigma^2$

Similarly $E\{ z(n) u(n-1) \} = \beta \sigma^2$

$p = \begin{bmatrix} 2\sigma^2 \\ \alpha\sigma^2 + \beta\sigma^2 \end{bmatrix}$

$\underline{h}_{opt} = R^{-1} p = \begin{bmatrix} 2 \\ \alpha + \beta \end{bmatrix}$

(b) RLS cost function

$$J(\underline{h}, n) = \sum_{k=0}^{n} \left(y(k) - \underline{h}^T \underline{u}(k) \right)^2 \lambda^{n-k}$$

RLS solution at time n is the minimiser of $J(\underline{h}, n)$. Call this $\underline{h}(n)$. Then

Show steps to obtain these

$$\left\{ \begin{aligned} \underline{h}(n) &= R^{-1}(n)\, p(n) \quad \text{where} \\ R(n) &= \lambda R(n-1) + \underline{u}(n) \underline{u}(n)^T \\ p(n) &= \lambda p(n) + y(n) \underline{u}(n)^T \end{aligned} \right.$$

For $\lambda = 0$, $\underline{h}(n) \longrightarrow h_{opt}$

(Can implement using matrix inversion lemma as in course notes.)

(c)

$$y(n) = x(n)\, z(n)$$

$$x(n) = \sum_{i=0}^{\infty} \alpha^i u(n-i) = u(n) + \alpha u(n-1) + \cdots$$

$$z(n) = \sum_{i=0}^{\infty} \beta^i u(n-i) = u(n) + \beta u(n-1) + \cdots$$

$$y(n) = u(n)^2 + 1 \cdot (\alpha + \beta) u(n) u(n-1) + \cdots$$

c.ii

Let $\underline{u}(n) = \begin{bmatrix} u(n)^2 \\ u(n)u(n-1) \end{bmatrix}$

As in part (a), minimiser is

$R^{-1} P$

$$E\{\underline{u}(n)\,\underline{u}(n)^T\} = \begin{bmatrix} E\{u(n)^4\} & 0 \\ 0 & E\{u(n)^4\} \end{bmatrix}$$

$$= \sigma^4 \underline{I}$$

$$P = \begin{bmatrix} E\{y(n)\,u(n)^2\} \\ E\{y(n)\,u(n)u(n-1)\} \end{bmatrix}$$

Given $E\{y(n)u(n)^2\} = \dfrac{\sigma^4}{1-\alpha\beta}$

$$y(n) = \alpha\beta\, y(n-1) + u(n)^2 + \alpha\, x(n-1)u(n) + \beta\, z(n-1)u(n)$$

$E\{y(n-1)\,u(n)u(n-1)\} = 0$

$E\{u(n)^2\,u(n)\,u(n-1)\} = 0$

$E\{x(n-1)\,u(n-1)\,u(n)^2\} = E\{x(n-1)u(n-1)\}\,E\{u(n)^2\}$
$= \sigma^2 \times \sigma^2$

$$E\{y(n)u(n)u(n-1)\}$$
$$= \alpha\sigma^4 + \beta\sigma^4$$

$$p = \sigma^4 \begin{bmatrix} \frac{1}{1-\alpha\beta} \\ \alpha+\beta \end{bmatrix}$$

$$h_{opt} = \frac{p}{\sigma^4}$$

Question 3

(a) $a_1 = 0 \Rightarrow x_n = a_2 x_{n-2} + w_n$

$$E\{x_n\} = a_2 E\{x_{n-2}\}$$

since in general $a_2 \neq 0$, $E\{x_n\} = 0$ for all n.

$$\therefore E\{x_{n+k} x_n\} = a_2 E\{x_{n+k-2} x_n\} + E\{w_{n+k} x_n\}$$

$$R_{xx}[k] = a_2 R_{xx}[k-2] + \begin{cases} 0 & k > 0 \\ \sigma^2 & k = 0 \end{cases}$$

$$x_n^2 = a_2^2 x_{n-2}^2 + w_n^2 + 2 a_2 x_{n-2} w_n$$

$$E\{x_n^2\} = a_2^2 E\{x_{n-2}^2\} + \sigma^2$$

This is the main part of answer

$$R_{xx}[0] = a_2^2 R_{xx}[0] + \sigma^2$$

$$R_{xx}[0] = \frac{\sigma^2}{1 - a_2^2}$$

$$\boxed{|a_2| < 1}$$ For finite variance

(b)

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + w_n$$

$$x_{n+k} = a_1 x_{n+k-1} + a_2 x_{n+k-2} + w_{n+k}$$

$$x_{n+k} x_n = a_1 x_{n+k-1} x_n + a_2 x_{n+k-2} x_n + w_{n+k} x_n$$

$$R_{xx}[k] = a_1 R_{xx}[k-1] + a_2 R_{xx}[k-2] + \begin{cases} \sigma_2 & k = 0 \\ 0 & k > 0 \end{cases}$$

$$r_0 = a_1 r_{-1} + a_2 r_{-2} + \sigma^2$$

$$r_1 = a_1 r_0 + a_2 r_{-1}$$

$$r_2 = a_1 r_1 + a_2 r_0$$

(c) $\hat{r}_0 = 1$, $\hat{r}_{2k} = 0.8^{|k|}$

(i) $r_0 = a_1 \times 0 + a_2 \times 0.8 + \sigma^2$

$0 = a_1 r_0 + a_2 \times 0$

$r_2 = 0.8 = a_2 r_0 = a_2$

Thus

$1 = 0.8\, a_2 + \sigma^2$

$\underline{0 = a_1}$

$\underline{a_2 = 0.8}$

$\underline{\sigma^2 = 1 - 0.64}$

(ii) $$S_x(e^{j\omega}) = \frac{\sigma^2}{|1 - a_1 e^{j\omega} - a_2 e^{j2\omega}|^2}$$

$$= \frac{0.36}{|1 - 0.8 e^{-j2\omega}|^2}$$

(iii) $$S_x(e^{j\omega}) = \sum_{k=-\infty}^{\infty} R_{xx}[k] e^{-j\omega k}$$

For $N = \infty$ $$\hat{S}_x = \sum_{k=-\infty}^{\infty} 0.8^{|k|} e^{-j2\omega k}$$

$$= \sum_{k=0}^{\infty} \left(0.8 e^{-j2\omega}\right)^k + \sum_{k=0}^{\infty} \left(0.8 e^{j2\omega}\right)^k - 1$$

$$= \frac{1}{1 - 0.8e^{-j2\omega}} + \frac{1}{1 - 0.8e^{j2\omega}} - 1$$

$$= \frac{1}{1 - 0.8e^{-j2\omega}} + \frac{1 - 1 + 0.8e^{j2\omega}}{1 - 0.8e^{j2\omega}}$$

$$= \frac{1 - 0.8e^{j2\omega} + 0.8e^{j2\omega} - 0.64}{|1 - 0.8e^{j2\omega}|^2}$$

$$= \frac{0.36}{|1 - 0.8e^{j2\omega}|^2}$$

Note that $N = \infty$ PSD is the same as previous section.

For finite N

$$\hat{S}_x = \sum_{k=-\infty}^{\infty} 0.8^{|k|} e^{-j2\omega k} W_{2k}$$

where W_k is a window function

where $W_{2k} = \begin{cases} 1 & |2k| < N \\ 0 & |2k| > N \end{cases}$

Thus $\hat{S}_x = \underbrace{\hat{S}_x}_{\text{computed for } N=\infty} \underset{\text{convolve}}{*} \underbrace{W(e^{j\omega})}_{\text{DTFT of window}}$

Let $N-1$ be even. You can write

$$\hat{S}_X = \sum_{k=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} 0.8^{|k|} e^{-j2\omega k}$$

This expression can be simplified using result for partial sum of a geometric series. Please show steps. Then check that your answer coincides with the $N=\infty$ solution in the limit as $N \to \infty$

(iv) Thus finite N PSD is

$$\underbrace{\hat{S}_X}_{\text{computed for } N=\infty} \underset{\text{Convolve}}{\overset{}{*}} \underbrace{W(e^{j\omega})}_{\text{DTFT of window}}$$

Large N implies $W(e^{j\omega}) \approx$ delta function and thus gives almost the same answer as $N=\infty$.

For smaller values of N the window will distort the PSD through its broader main lobe and more significant side loabs.

Question 4

(a) $w_{n+k} x_n = a_1 w_{n+k} x_{n-1} + a_2 w_{n+k} x_{n-2} + b_0 w_{n+k} w_n + b_1 w_{n+k} w_{n-1}$

$k=-1$

$w_{n-1} x_n = a_1 w_{n-1} x_{n-1} + a_2 w_{n-1} x_{n-2} + b_0 w_{n-1} w_n + b_1 w_{n-1}^2$

$E\{w_{n-1} x_n\} = b_0 a_1 + b_1$

because $E\{w_{n-1} x_{n-1}\} = b_0$

$E\{w_{n-1} x_{n-2}\} = 0$

$E\{w_{n-1} w_n\} = 0$

$k=0$

$w_n x_n = a_1 x_{n-1} w_n + a_2 x_{n-2} w_n + b_0 w_n^2 + b_1 w_{n-1} w_n$

$E\{w_n x_n\} = a_1 \times 0 + a_2 \times 0 + b_0 \times 1 + b_1 \times 0$

$E\{w_n x_n\} = b_0$

$k=1$

$E\{w_{n+1} x_n\} = 0$ because x_n has no w_{n+1} term

Similarly $k>1$, $E\{w_{n+k} x_n\} = 0$

(b) $x_{n+k} \times x_n = (a_1 x_{n+k-1} + a_2 x_{n+k-2} + b_0 w_{n+k} + b_1 w_{n+k-1}) \times x_n$

$R_{xx}[k] = a_1 R_{xx}[k-1] + a_2 R_{xx}[k-2] + b_0 R_{wx}[k] + b_1 R_{wx}[k-1]$

Let $r_k = R_{xx}[k]$

$r_0 = a_1 r_1 + a_2 r_2 + b_0 \times b_0 + b_1 (b_0 a_1 + b_1)$

$$r_1 = a_1 r_0 + a_2 r_1 + b_0 \times 0 + b_1 b_0$$

$$\underline{r_1 = a_1 r_0 + a_2 r_1 + b_1 b_0}$$

$$\underline{r_2 = a_1 r_1 + a_2 r_0}$$

$$\underline{r_3 = a_1 r_2 + a_2 r_1}$$

$$\vdots$$

(c) From previous part

$$r_2 = a_1 r_1 + a_2 r_0$$
$$r_3 = a_1 r_2 + a_2 r_1$$

$$\begin{bmatrix} r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} r_1 & r_0 \\ r_2 & r_1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} r_1 & r_0 \\ r_2 & r_1 \end{bmatrix}^{-1} \begin{bmatrix} r_2 \\ r_3 \end{bmatrix}$$

(d) Using eqns for r_0 and r_1 gives

$$r_0 - a_1 r_1 - a_2 r_2 = b_0^2 + b_1^2 + b_1 b_0 a_1 \quad \text{— } ①$$
$$r_1 - a_1 r_0 - a_2 r_1 = b_1 b_0 \quad \text{— } ②$$

Write $b_1 = \dfrac{r_1 - a_1 r_0 - a_2 r_1}{b_0}$

and substitute ② into ① to get an equation only in terms of b_0 to obtain an equation of the form

$$b_0^2 + \frac{c}{b_0^2} = d$$ for some constants c and d

Now solve to get two possible values for b_0^2 and then solve for b_1

Alternatively, once we have solved for a_1 and a_2, we have

$$y_n = b_0 w_n + b_1 w_{n-1}$$

where $y_n = x_n - a_1 x_{n-1} - a_2 x_{n-2}$.

and we can use the spectral factorisation method using $R_{YY}[-1]$, $R_{YY}[0]$, $R_{YY}[1]$

(e) We are given

$$r_k = \sum_{p=1}^{P} a_p r_{k-p} + \sum_{q=0}^{Q} b_q h_{q-k}$$

We see that

$$\sum_{q=0}^{Q} b_q h_{q-k} = 0 \quad \text{for} \quad k > Q$$

since the impulse response $h_n = 0$ for $n < 0$

so

$$r_k = \sum_{p=1}^{P} a_p r_{k-p} \quad \text{for} \quad k = Q+1, Q+2, \ldots$$

We can use P such values to get P equations for the P unknowns $(a_1, \ldots, a_P)$.

Thus we need $r_{Q+1}, \ldots, r_{Q+P}$.

Having solved for $(a_1, \ldots, a_P)$, we then use, in addition, $r_0, r_1, \ldots, r_Q$ to solve for $(b_0, \ldots, b_Q)$ using spectral factorisation.

We can write

$$x_k = \sum_{p=1}^{P} a_p x_{k-p} + \sum_{q=0}^{Q} b_q W_{k-q}$$

as

$$y_k = \sum_{q=0}^{Q} b_q W_{k-q}$$

$$y_k = x_k - \sum_{p=1}^{P} a_p x_{k-p}$$

and y_k is now an MA process.

To solve for $b_0, \dots, b_Q$ using spectral factorisation we need

$$R_{yy}[0], \dots, R_{yy}[Q]$$

Write

$$y_k = a^T \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-P} \end{bmatrix}$$

$R_{yy}[0]$ needs $R_{xx}[0], \dots, R_{xx}[P]$

$R_{yy}[1]$ needs $R_{xx}[0], \dots, R_{xx}[P+1]$

$\vdots$

Q1 Least Mean Square for removing Intersymbol Interference

The most popular question, largely done well with the exception of part b.ii. This was a simple MA process and its autocorrelation should have been familiar to all. Unfortunately part b.iii similarly done poorly.

Q2 System Identification

Least popular question but turned out to be a good mark earner for those who attempted it. The description of the RLS in part b done poorly by the majority. It was satisfying to see part c.ii being solved by directly differentiating the cost function to find the stationary points, rather than expanding the cost function (expectation operator) and then differentiating which would have been considerably more work.

Q3 Model order and Power Spectrum Estimation

Part c.iii was surprisingly done poorly, for both the finite N and infinite N case; this was a simple calculation involving a geometric series and many either failed to see this or got the details wrong.

Q4 Fitting an ARMA(P,Q) model to data

A very popular question but done poorly on average. Parts a and b were routine calculations that would have been familiar to Part 2A students but surprisingly done quite badly. Many failed to see how part c could have been solved independently of the MA coefficients. The use of Spectral Factorisation for part d unnecessarily complicated for only 2 MA coefficients. A direct solution would have been better. Part e was not answered in its intended generality and those that did failed to justify their assertions properly.

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