(4 M12, 2015) Dr J Li

Question 1

(a) $\nabla^2 u = -\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right]$ (spherical polars)

$= 0$ for $\underline{x} \neq 0$

For $R > 0$,

$$\nabla u = \frac{1}{4\pi}\frac{1}{r^2}\underline{e}_r \Rightarrow \oint_{S_R} \nabla u \cdot \underline{ds} = \frac{1}{4\pi}\oint_{S_R}\frac{1}{r^2}ds = 1$$

Hence

$$\iiint_{R^3} \nabla^2 u \, dV = \underbrace{\iiint_{\mathbb{R}^3 - V_R} \nabla^2 u \, dV}_{=0} + \underbrace{\iiint_{V_R} \nabla^2 u \, dV}_{=\oint_{S_R} \nabla u \cdot \underline{ds}} = 1$$

← (Gauss)

This is what's required to satisfy (1).

(b) $\nabla \times \underline{B} = \mu \underline{J} = \nabla \times \nabla \times \underline{A}$

From Data Book, $\nabla^2 \underline{A} = \nabla(\underbrace{\nabla \cdot \underline{A}}_{=0}) - \nabla \times \nabla \times \underline{A} = -\nabla \times \nabla \times \underline{A}$

$$\Rightarrow \nabla^2 \underline{A} = -\mu \underline{J}$$

From (a) solutions to δ function at $\underline{x}'$

$$\nabla^2 u = \delta(\underline{x}'),$$

is

$$u(\underline{x}) = -\frac{1}{4\pi}\frac{1}{|\underline{x} - \underline{x}'|}$$

If the δ-function has strength S',

$$\nabla^2 u = S'\delta(\underline{x}')$$

$$u(\underline{x}) = -\frac{1}{4\pi}\frac{S'}{|\underline{x} - \underline{x}'|}$$

Question 1

Using superposition, the solutions to Poisson Eqn

$$\nabla^2 u = S(\underline{x})$$

is
$$u(\underline{x}) = -\frac{1}{4\pi}\int \frac{S(\underline{x}')\,dV'}{|\underline{x}-\underline{x}'|}$$

Applying this to each of components of ③ yields

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\,dV'$$

set $r = |\underline{x}-\underline{x}'|$, $\underline{r} = \underline{x}-\underline{x}'$ (Spherical Polars)

(c)
$$\nabla\left(\frac{1}{|\underline{x}-\underline{x}'|}\right) = \nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\underline{e}_r = -\frac{\underline{e}_r}{r^2} = -\frac{\underline{r}}{r^3}$$

Hence
$$\underline{B} = \nabla\times\underline{A} = \frac{\mu_0}{4\pi}\int \nabla\times\left[\frac{\underline{J}(\underline{x}')}{|\underline{x}-\underline{x}'|}\right]dV'$$

$$= \frac{\mu_0}{4\pi}\int \nabla\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)\times\underline{J}(\underline{x}') + \frac{1}{|\underline{x}-\underline{x}'|}\underbrace{\nabla\times\underline{J}(\underline{x}')}_{=0}\,dV'$$

$$= \frac{\mu_0}{4\pi}\int -\frac{\underline{x}-\underline{x}'}{|\underline{x}-\underline{x}'|^3}\times\underline{J}(\underline{x}')\,dV'$$

$$= -\frac{\mu_0}{4\pi}\int \frac{\underline{r}\times\underline{J}(\underline{x}')}{|\underline{r}|^3}\,dV'.$$

(Biot - Savart law)

Comments

(1) Few Candidates showed $\iiint_{\mathbb{R}^3}\nabla^2 u\,dV = 1$.

(2) Few candidates gave the details on

$$\nabla\left(\frac{1}{|\underline{x}-\underline{x}'|}\right) = -\frac{\underline{x}-\underline{x}'}{|\underline{x}-\underline{x}'|^3}$$

Question 2

(a) Assume $\eta = e^{i(kx-\omega t)}$

$$\frac{\partial \eta}{\partial x} = ik\eta, \quad \frac{\partial^4 \eta}{\partial x^4} = (ik)^4 \eta = k^4 \eta$$

$$\frac{\partial \eta}{\partial t} = i\omega \eta, \quad \frac{\partial^2 \eta}{\partial t^2} = (i\omega)^2 \eta = -\omega^2 \eta$$

$$\Rightarrow \quad IEk^4 + S - \rho A\omega^2 = 0$$

$$\omega^2 = \frac{IEk^4 + S}{\rho A}$$

$$2\omega \frac{d\omega}{dk} = \frac{4IEk^3}{\rho A} \Rightarrow c_g = \frac{d\omega}{dk} = \frac{2IEk^3}{\rho A\omega}$$

But $c_p = \omega/k \Rightarrow \dfrac{c_g}{c_p} = \dfrac{2IEk^4}{\rho A\omega^2} = \boxed{\dfrac{2IEk^4}{IEk^4+S}}$

(b) $c_g = c_p \Rightarrow IEk^4 + S = 2IEk^4 \Rightarrow k^4 = \dfrac{S}{IE}$

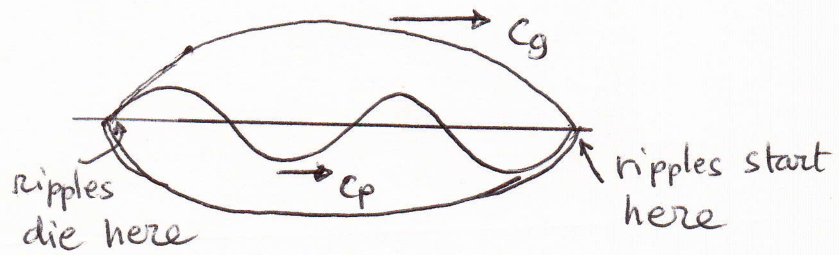
$$\Rightarrow k_0 = \left(\frac{S}{IE}\right)^{\frac{1}{4}}$$

Furthermore, $k \gtrless k_0 = \left(\frac{S}{IE}\right)^{\frac{1}{4}} \Rightarrow IEk^4 \gtrless S \Rightarrow 2IEk^4 \gtrless IEk^4 + S$

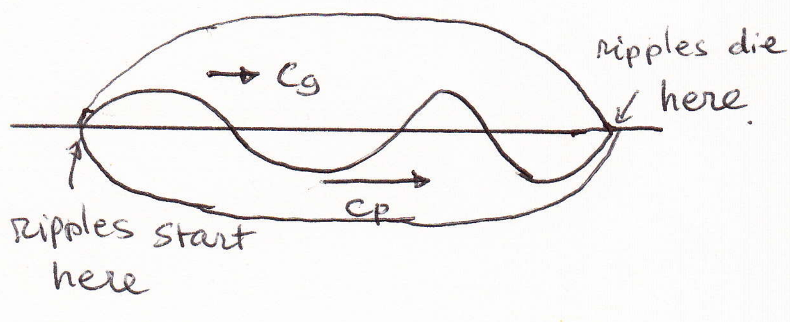
$$\Rightarrow \frac{c_g}{c_p} = \frac{2IEk^4}{IEk^4+S} \gtrless 1 \Rightarrow c_g \gtrless c_p.$$

Hence

$k > k_0$, $c_g > c_p$



$k < k_0$, $c_g < c_p$



Question 2

(c) Assume $\eta = e^{i(\underline{k}\cdot\underline{x} - \omega t)} = e^{i(k_x x + k_y y - \omega t)}$

$k = |\underline{k}| = (k_x^2 + k_y^2)^{\frac{1}{2}}$, $\underline{k} = (k_x, k_y)$

$$\nabla^2 \eta = \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)\eta = [(ik_x)^2 + (ik_y)^2]\eta = -(k_x^2 + k_y^2)\eta$$

$$\nabla^4 \eta = \nabla^2(\nabla^2 \eta) = (-k_x^2 - k_y^2)\nabla^2\eta = (k_x^2 + k_y^2)^2 \eta = k^4 \eta$$

$\partial^2\eta/\partial t^2 = -\omega^2 \eta.$

$$\Rightarrow \quad D k^4 + S - \rho A \omega^2 = 0$$

$$\Rightarrow \quad \omega^2 = \frac{Dk^4 + S}{\rho A}$$

$$\Rightarrow \quad 2\omega \nabla \omega = \frac{4Dk^3 \nabla k}{\rho A} = \frac{4Dk^2 \underline{k}}{\rho A}$$

as $\nabla k = \left(\frac{\partial k}{\partial x}, \frac{\partial k}{\partial y}\right) = \frac{1}{2}\frac{1}{k}(2k_x, 2k_y) = \frac{\underline{k}}{k}$

Hence $c_p = \frac{\omega}{k} \Rightarrow \frac{\underline{c_g}}{c_p} = \frac{\nabla \omega}{c_p} = \frac{2Dk^3 \underline{k}}{\rho A \omega^2} = \underline{\underline{\frac{2Dk^3 \underline{k}}{Dk^4 + S}}}$

$\underline{c_g}$ is // to $\underline{k}$ for flexural vibration.

$\underline{c_g}$ is $\perp$ to $\underline{k}$ for rapidly-rotating fluid or initial waves.

Comments

$\boxed{\begin{array}{l} \text{(1) Few candidates showed } k \gtrsim k_0 \;\; c_g \gtrless g \\ \text{though they used it.} \\ \text{(2) Many students did not give the details} \\ \text{of their calculations of } \nabla^4\eta \text{ and } \nabla k. \end{array}}$

Question 3

(a) The augmented integrand is

$$\tilde{F} = (u')^2 + \Lambda(u^2 - 1)$$

where Λ is the constant Lagrangian multiplier to be determined.

Augmented functional

$$\tilde{I}[u] = \int_0^1 (u')^2 + \Lambda(u^2-1)\, dx$$

(b) $$\frac{\partial \tilde{F}}{\partial u} = 2\Lambda u \quad , \quad \frac{\partial \tilde{F}}{\partial u'} = 2u'$$

The Euler Eqn. for $\tilde{F} = \tilde{F}(\Lambda, x, u, u')$ is

$$\frac{\partial \tilde{F}}{\partial u} - \frac{d}{dx}\left(\frac{\partial \tilde{F}}{\partial u'}\right) = 0 ,$$

or $$2\Lambda u - (2u')' = 0.$$

$$u'' - \Lambda u = 0.$$

(c) The boundary condition at $x = 1$

is $\frac{\partial \tilde{F}}{\partial u'}(1) = 0$, that is $u'(1) = 0$.

(d) There are three cases $\Lambda > 0$, $\Lambda = 0$, $\Lambda < 0$.

① For $\Lambda > 0$, set $\Lambda = k^2$, $k>0$ then

$$u'' - k^2 u = 0 ,$$ the general solution

is $$u = A \sinh(kx) + B \cosh(kx)$$

boundary condition $u(0)=0 \Rightarrow B\cosh(\cancel{kx})=0 \Rightarrow B=0$

boundary condition $u'(1)=0 \Rightarrow Ak\cosh(k)=0$

$\Rightarrow A=0$ (as $k \cancel{>} 0$) $\Rightarrow \cancel{A}=0$, $u=0$.

but $u=0$ does not satisfy the constrant

$\int_0^1 u^2\,dx=1$, so there is not solution

for $\cancel{k}>0$ $(\Lambda>0)$.

2 For $\Lambda=0$, $u''=0$, $u=ax+b$

boundary condition $u(0)=0 \Rightarrow b=0$
$u'(1)=0 \Rightarrow a=0$ } $u=0$, but does not satisfy the constraint,

3 For $\Lambda<0$, $\Lambda=-k^2$, $k>0$.

$u''+k^2u=0 \Rightarrow$ general solution $u=A\sin(kx)+B\cos(kx)$

B.C. $u(0)=0 \Rightarrow B=0$, $u=A\sin(kx)$, $u'=Ak\cos(kx)$

B.C $u'(1)=0 \Rightarrow \cos(k\cancel{x})=0 \Rightarrow k=\frac{n\pi}{2}$, $n\in\mathbb{Z}$, n is odd. ~~n≠0~~

To satisfy the constraint $\int_0^1 u^2dx=1$,

$$\int_0^1 A^2\sin^2(kx)\,dx = A^2\int_0^1 \frac{1-\cos(2kx)}{2}\,dx$$

As $n\neq 0$ $$= \frac{A^2}{2}\left[1-\frac{\sin(n\pi x)}{n\pi}\right]_0^1 = \frac{A^2}{2}=1 \Rightarrow A=\sqrt{2}$$

~~if n=0~~

Finally, the stationary function is

$$\boxed{u(x)=\sqrt{2}\sin\left(\frac{n\pi}{2}x\right) \text{ for } n\in\mathbb{Z},\ n \text{ is odd.}}$$

<u>Comment</u>: Many student could not solve this simple ODE!

Question 4

: <u>Comment</u> There was a similar but more difficult question last year. It is very surprising that fewer candidates attempted this one this year.

(a)
$$\frac{d}{dx}\left(x\frac{du}{dx}\right) = -1 \qquad ①$$

multiply by v with $v(1)=0$, & integrate over $[1,2]$

$$\int_1^2 \frac{d}{dx}\left(x\frac{du}{dx}\right) v\,dx = -\int v\,dx$$

Integration by part

$$x\frac{du}{dx}v\Big|_1^2 - \int_1^2 x\frac{du}{dx}\frac{dv}{dx}\,dx = -\int v\,dx$$

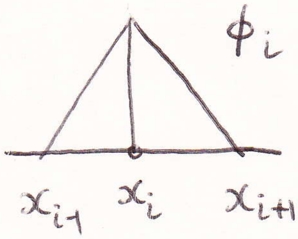
weak form $\displaystyle\int_1^2 x\frac{du}{dx}\frac{dv}{dx}\,dx = \int v\,dx$

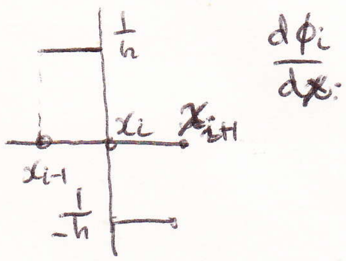
(b) Variational form

$$I[u] = \int_1^2 \left[\frac{1}{2}x\left(\frac{du}{dx}\right)^2 - u\right]dx.$$

$0<i<n$

(c)





$0<i<n$

① for $j=i$, $\displaystyle\int_1^2 x\frac{d\phi_i}{dx}\cdot\frac{d\phi_i}{dx}\,dx = \frac{1}{h^2}\int_{x_{i-1}}^{x_{i+1}} x\,dx = \frac{x_{i+1}^2 - x_{i-1}^2}{2h^2}$

② $j=i-1$, $\displaystyle\int_1^2 x\frac{d\phi_i}{dx}\cdot\frac{d\phi_{i-1}}{dx}\,dx = -\frac{1}{h^2}\int_{x_{i-1}}^{x_i} x\,dx = -\frac{x_i^2 - x_{i-1}^2}{2h^2}$

③ $j=i+1$, $\displaystyle\int_1^2 x\frac{d\phi_i}{dx}\frac{d\phi_{i+1}}{dx}\,dx = -\frac{1}{h^2}\int_{x_i}^{x_{i+1}} x\,dx = -\frac{x_{i+1}^2 - x_i^2}{2h^2}$

⑤ otherwise $k_{ij}=0$ because there is no overlap between (x_{i-1}, x_{i+1}) and $[x_{j-1}, x_{j+1}]$.

④ $K_{00} = \dfrac{x_1^2 - x_0^2}{2h^2}$, $K_{nn} = \dfrac{x_n^2 - x_{n-1}^2}{2h^2}$

(2)

(d) $0 < i < n$ $\int_1^2 \phi_i \, dx = \int_{x_{i-1}}^{x_{i+1}} \phi_i \, dx = \frac{1}{2} \times 1 \times 2h = h$

Because $u(1) = 0$, $u(2) = -1$,

these boundary conditions impose $c_0 = 0$, $c_2 = -1$.

For c_1, take $i = 1$, Galerkin method gives

$$K_{10} \overset{0}{\overset{\shortparallel}{c_0}} + K_{11} c_1 + K_{12} \overset{-1}{\overset{\shortparallel}{c_2}} = \int_1^2 \phi_i \, dx = h$$

for $n = 2$, $h = \frac{1}{2}$, $x_0 = 1$, $x_1 = 3/2$, $x_2 = 2$.

Hence,

$$\frac{2^2 - 1}{2 \times (\frac{1}{2})^2} c_1 - \frac{2^2 - (3/2)^2}{2 \times (\frac{1}{2})^2} \overset{-1}{\overset{\shortparallel}{c_2}} = \frac{1}{2}$$

$$6 c_1 - \frac{7}{2}(-1) = \frac{1}{2}$$

$$c_1 = -\frac{1}{2}$$

(e) The ODE: $\frac{d}{dx}\left(x \frac{du}{dx}\right) = -1$,

$$x \frac{du}{dx} = -x + A$$

$$\frac{du}{dx} = -1 + \frac{A}{x}$$

$$\Rightarrow u = -x + A \ln x + B, \quad (A, B \text{ two constants})$$

$u(1) = 0$, $u(2) = -1 \Rightarrow A = 0, B = 1$, so $\boxed{u = -x + 1}$

$u(3/2) = -\frac{1}{2} = \bar{u}(\frac{3}{2})$, the exact & approximate solutions are the same.

<u>Explanation</u>: The approximate solution space consists of all piecewise linear functions. As the exact solution is a linear function, it is not surprising that the approximate solution is the exact solution, as it is the best approximation in the approximate space.