 4PP112

(1-1)

(a) In dispersive waves the phase speed is a function of wavenumber, whereas non-dispersive waves have a phase speed indepedent of wavenumber. In non-dispersive systems all Fourier modes travel at the same speed. So if an initial disturbance is Fourier decomposed into many modes, each mode travels travels at the same speed and so each mode travels the same distance in a given time. Thus, in the absence of dissipation, disturbances travel without change shape in a non-dispersive system. In dispersive system, on the other hand, disturbances change shape (spread out) as they propagate.

(b) $$\eta(x,t) = A(x,t)\exp[i\theta(x\,t)]$$

$$\frac{\partial k}{\partial t} = \frac{\partial^2\theta}{\partial t\partial x} = \frac{\partial^2\theta}{\partial x\partial t} = -\frac{\partial\omega}{\partial x} = -\frac{d\omega}{dk}\frac{\partial k}{\partial x}$$

Thus k satisfies the ODE

$$\frac{\partial k}{\partial t} + C_g(k)\frac{\partial k}{\partial x} = 0, \quad C_g = \frac{d\omega}{dk}$$

Solution is

$$k = f(x - C_g t), \text{ for some } f(\cdot)$$

Check, $\frac{\partial k}{\partial t} = -C_g f'$, $\frac{\partial k}{\partial x} = f' \Rightarrow \frac{\partial k}{\partial t} = -C_g\frac{\partial k}{\partial x}$

If $x = C_g t$ (i.e. travel at C_g), then k is constant

(c)

$$\frac{\partial^2}{\partial t^2} \nabla^2 u_z + N^2 \frac{\partial^2 u_z}{\partial x^2} = 0$$

$\underline{k} = (k_x, k_z)$

Trial solution: $u_z = \hat{u} \exp[i(\underline{k} \cdot \underline{x} - \omega t)]$

$$\Rightarrow (-k^2)(-\omega^2) + N^2(-k_x^2) = 0$$

$$\Rightarrow \omega^2 = N^2 \frac{k_x^2}{k_x^2 + k_z^2}$$

$$\Rightarrow \omega = \pm N \frac{k_x}{\sqrt{k_x^2 + k_z^2}}$$

$\Rightarrow$ realisable frequency

$$0 \leq |\omega| \leq N$$

(d)

$$[c_g]_z = \frac{\partial \omega}{\partial k_z} = \pm N \left(-\frac{1}{2}\right)(2k_z) \frac{k_x}{|\underline{k}|^3}$$

$$= \mp N \frac{k_x k_z}{|\underline{k}|^3}$$

$$[c_g]_x = \frac{\partial \omega}{\partial k_x} = \pm N \left[\frac{1}{|\underline{k}|} - \frac{k_x^2}{|\underline{k}|^3} \right]$$

$$= \pm N \frac{k_z^2}{|\underline{k}|^3}$$

$$\Rightarrow \underline{c_g} = \left[\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_z} \right] = \pm \frac{N k_z}{|\underline{k}|^3} \left[k_z \hat{\underline{e}}_x - k_x \hat{\underline{e}}_z \right].$$

(a) In steady state

$$\alpha \nabla \cdot (\nabla T) = -\frac{\dot{Q}}{\rho C_p} \delta$$

Integrate over a sphere centered at the origin

$$\alpha \int_V \nabla \cdot (\nabla T) dV = \alpha \oint_S \nabla T \cdot \underline{ds} = -\frac{\dot{Q}}{\rho C_p} \int \delta dV$$

$$\Rightarrow \quad \oint \nabla T \cdot \underline{ds} = -\frac{\dot{Q}}{\rho C_p \alpha}$$

T is spherically symmetric, so

$$\frac{\partial T}{\partial r} \pi r^2 = -\frac{\dot{Q}}{\rho C_p \alpha}$$

$$\Rightarrow \quad \frac{\partial T}{\partial r} = -\frac{\dot{Q}}{\rho C_p \alpha} \frac{1}{4\pi r^2} \quad \text{Green function}$$

$$\Rightarrow \quad T = \frac{Q}{\rho C_p \alpha} \frac{1}{4\pi r} + \cancel{\text{const}}$$

(b)

$$\nabla^2 T = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{\Gamma}{r}\right)\right)$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left[r \frac{\partial \Gamma}{\partial r} - \Gamma\right] = \frac{1}{r^2}\left[r \frac{\partial^2 \Gamma}{\partial r^2} + \cancel{\frac{\partial \Gamma}{\partial r}} - \cancel{\frac{\partial \Gamma}{\partial r}}\right]$$

Thus $\frac{\partial T}{\partial t} = \alpha \nabla^2 T$ becomes $\frac{\partial}{\partial t}\left(\frac{\Gamma}{r}\right) = \alpha\left(\frac{1}{r} \frac{\partial^2 \Gamma}{\partial r^2}\right) \Rightarrow \frac{\partial \Gamma}{\partial t} = \alpha \frac{\partial^2 \Gamma}{\partial r^2}$

Now let $\Gamma = F\left(\frac{r}{\sqrt{\alpha t}}\right) = \Gamma(\eta)$, $\eta = \frac{r}{\sqrt{\alpha t}}$

$$\frac{\partial \Gamma}{\partial t} = -\frac{1}{2} \frac{r}{\sqrt{\alpha t^3}} F'(\eta) = -\frac{1}{2t} \eta F'(\eta)$$

$$\frac{\partial^2 \Gamma}{\partial r^2} = \left(\frac{1}{\sqrt{\alpha t}}\right)^2 F''(\eta) = \frac{1}{\alpha t} F''(\eta)$$

substitute into P.D.E.

$$\frac{\partial \Gamma}{\partial t} = -\frac{1}{2t}\eta F'(\eta) = \alpha \frac{\partial^2 \Gamma}{\partial r^2} = \alpha \frac{1}{\alpha t} F''(\eta)$$

$$\Rightarrow \quad F''(\eta) + \tfrac{1}{2}\eta F'(\eta) = 0 \ , \quad F''/F' = -\frac{\eta}{2} = [\ln(F')]$$

(c) Integrate once: $F'(\eta) = A \exp[-\eta^2/4]$, $A = \text{const}$

Integrate again

$$F(\eta) = A \int_0^\eta \exp[-\eta^2/4] d\eta + B$$

$$\Rightarrow \quad rT = A \int_0^\eta e^{-\eta^2/4} dy + B$$

But $rT \to 0$ for $r \to \infty$ $\Rightarrow$ $A\sqrt{\pi} + B = 0$

$$\Rightarrow \quad rT = A\left[\int_0^\eta e^{-\eta^2/4} dy - \sqrt{\pi}\right]$$

To find A, we match this to solution in (a)

For $\eta \ll 1$, $T = -\frac{A\sqrt{\pi}}{r} = \frac{\dot{Q}}{\rho C_p \alpha}\frac{1}{4\pi r}$

$$\Rightarrow \quad A = -\frac{\dot{Q}}{\rho C_p \alpha}\frac{1}{4\pi^{3/2}}$$

$$\Rightarrow \quad T = \frac{\dot{Q}}{\rho C_p \alpha}\frac{1}{4\pi r}\left[1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2/4} d\eta\right]$$

(a) The curve length

$$I(y) = \int_{-1}^{1} \sqrt{1 + y'^2(x)}\, dx$$

(b) This is subject to the integral constraint

$$\int_{-1}^{1} y(x)\, dx = A$$

The augmented integrand is

$$F(\lambda, x, u, u') = \sqrt{1 + (y')^2} + \lambda y$$

(c) The Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \lambda - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

(d) Integrating once with respect to x

$$\frac{y'}{\sqrt{1+(y')^2}} = \lambda(x + A) \quad \text{where } C \text{ integration const}$$

$$\frac{(y')^2}{1+(y')^2} = \lambda^2 (x+A)^2$$

$$(y')^2\left(1 - \lambda^2 (x+A)^2\right) = \lambda^2 (x+A)^2$$

$$y' = \frac{\lambda(x+A)}{\sqrt{1 - \lambda^2 (x+A)^2}}$$

Integrating again with respect to x leads to

$$y = -\frac{1}{\lambda}\sqrt{1-\lambda^2(x+A)^2} - B$$

or

$$(x+A)^2 + (y+B)^2 = \frac{1}{\lambda^2}$$

which is the equation for a circular arc with center at $(-A, -B)$ and radius $\left|\frac{1}{\lambda}\right|$

(e) The integration constants A and B can be obtained from the BCs. $y(-1) = y(1) = 0$

$u(-1) = 0$: $(-1+A)^2 + B^2 = \frac{1}{\lambda^2}$ ①

$u(1) = 0$: $(1+A)^2 + B^2 = \frac{1}{\lambda^2}$ ②

① − ② $\Rightarrow$ $-4A = 0 \Rightarrow A = 0$

① + ② $\Rightarrow$ $B = \sqrt{\frac{1}{\lambda^2} - 1}$

For B to be real, $\frac{1}{\lambda^2} > 1 \Rightarrow |\lambda| \leq 1$.

This is consistent with the fact that there is no circular arc that passes through $(-1,0)$ and $(1,0)$ with radius less than one.

(a) (i) Let v be a differential defined in [0,1]

the dèrectional derivative at u(x)

$$DI[u](v) = \frac{d}{d\varepsilon} I[u+\varepsilon v] = \frac{d}{d\varepsilon}\int_0^1 F(x, u+\varepsilon v, u'+\varepsilon v', u''+\varepsilon v'')dx$$

$$= \int_0^1 \left(\frac{\partial F}{\partial u} v + \frac{\partial F}{\partial u'} v' + \frac{\partial F}{\partial u''} v''\right) dx$$

(ii) Integration by part,

$$DI[u](v) = \int_0^1 [\frac{\partial F}{\partial u} - \frac{d}{dx}(\frac{\partial F}{\partial u'})]v - \frac{d}{dx}(\frac{\partial F}{\partial u''})v' dx$$

$$+ \left(\frac{\partial F}{\partial u'} v + \frac{\partial F}{\partial u''} v'\right)\Big|_0^1$$

$$= \int_0^1 [\frac{\partial F}{\partial u} - \frac{d}{dx}(\frac{\partial F}{\partial u'}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial u''})] v \, dx$$

$$+ \left(\frac{\partial F}{\partial u'} v + \frac{\partial F}{\partial u''} v' - \frac{d}{dx}(\frac{\partial F}{\partial u''}) v\right)\Big|_0^1$$

If we choose v such that $v(0) = v'(0) = v(1) = v'(1) = 0$,

the boundary terms in the above equation disappear.

As u(x) is a stationary function of I[u],

$$DI[u](v) = \int_0^1 \left[\frac{\partial F}{\partial u} - \frac{d}{dx}(\frac{\partial F}{\partial u'}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial u''})\right] v \, dx = 0$$

for any function v such as $v(0) = v'(0) = v(1) = v'(1) = 0$

the fundamental lemma of the calculus implies

~~the integrand is zer~~

$$\boxed{\frac{\partial F}{\partial u} - \frac{d}{dx}(\frac{\partial F}{\partial u'}) + \frac{d^2}{dx^2}(\frac{\partial F}{\partial u''}) = 0}$$

the Euler-Lagrange Equation.

(a) (iii) Dirichlet boundary conditions require that u and u' take fixed values at $x=0$ and $x=1$.

Or, Natural boundary conditions require that

$$\boxed{\frac{\partial F}{\partial u'} - \frac{d}{dx}\left(\frac{\partial F}{\partial u''}\right) = 0} \text{ and } \boxed{\frac{\partial F}{\partial u''} = 0} \text{ at}$$

$x=0$ and $x=1$

(b) (i) the augmented integrand

$$\tilde{F} = \frac{(u'')^2}{2} + \lambda u$$

where λ is a parameter

(ii) Euler Eqn: $\frac{\partial \tilde{F}}{\partial u} - \frac{d}{dx}\left(\frac{\partial \tilde{F}}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \tilde{F}}{\partial u''}\right) = \lambda + u'''' = 0$

(iii) Integration of the above equation

$$u = Ax^4 + Bx^3 + Cx^2 + Dx + E \quad \text{where } A = -\frac{\lambda}{24}$$

$$u' = 4Ax^3 + 3Bx^2 + 2Cx + D$$

$$u(0) = u'(0) = 0 \Rightarrow D = E = 0$$

$$\left.\begin{aligned} u(1) = 0 &\Rightarrow A + B + C = 0 \\ u'(1) = 0 &\Rightarrow 4A + 3B + 2C = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} B &= -2A \\ C &= A \end{aligned}$$

Finally, $\int_0^1 u(x)\,dx = A\int_0^1 x^4 - 2x^3 + x^2\,dx$

$= \frac{A}{30} = 1$ (integral constraint)

$A = 30$ and $\boxed{u = 30x^4 - 60x^3 + 30x^2}$

Assessor's comments, Module 4M12: Partial differential equations and variational methods, 2016

Q1 PDE on group speed

Overall the performance was satisfactory although it was noticeable that very few candidates were able to derive the properties of group velocity using the phase-function argument. This is probably because, despite being covered in detail in lectures, it has not been examined before.

Q2 PDE on similarity solution

A straight forward question on spherically symmetric diffusion and self-similar solutions. This was attempted by all but one candidate. The responses were satisfactory although most candidates struggled with the first part of the question in which they has to integrate over a three-dimensional delta function.

Q3 Variational method on Constrain problem

It is a constrain problem. Most of students understand how to use the method of Lagragian multiplier. It is surprising that many students' skills for solving a simple ODE was not good enough to find the correct solutions.

Q4 Variational methods involving first and second derivatives

This question checks students' understanding of the principles of variational methods. It is good (I am pleased) to see that many candidates candidates can deduce the Euler-Lagrangian Equation for variational problems involving high derivatives. The difficult part is to deduce all the right boundary conditions for this equation.

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