4M12 2018 CRIB

(1) (a)

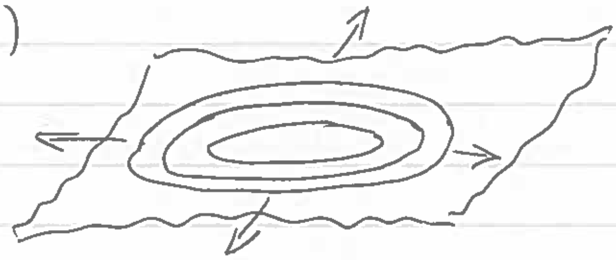
(i) Deep water $\omega^2 = gk \Rightarrow \begin{cases} \text{Phase speed} = \dfrac{\omega}{k} = \sqrt{g/k} \\ \text{Group speed} = \dfrac{\partial \omega}{\partial k} = \dfrac{1}{2}\sqrt{g/k} \end{cases}$

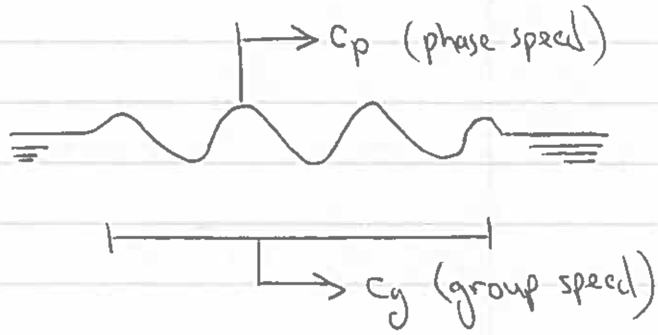
(dispersive)

Shallow water $\omega^2 = ghk^2 \Rightarrow \begin{cases} \text{Phase speed} = \dfrac{\omega}{k} = \sqrt{gh} \\ \text{Group speed} = \dfrac{\partial \omega}{\partial k} = \sqrt{gh} \end{cases}$

(non-dispersive)

(ii)





The wave packet travels at c_g ($= \frac{1}{2} c_p$) and the crests travel at c_p. New crests continually appear at the left of the packet, ripple across the packet, and then dissapear at the right of the wave packet.

(b) (i) Look for solution of the form $\eta = \exp(j(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t))$

$$k^4 D + S = \rho \varpi^2$$

$$\Rightarrow \quad \varpi^2 = \frac{Dk^4 + S}{\rho}$$

$$\Rightarrow \quad \underset{\sim}{c_g} = \frac{\partial \omega}{\partial k_i} = \frac{1}{2\omega}\frac{\partial \omega^2}{\partial k_i} = \frac{4Dk^2 \underset{\sim}{k}}{2\rho\omega}$$

$$\Rightarrow \quad \underline{\underset{\sim}{c_g} = \frac{2Dk^2}{\rho\omega}\underset{\sim}{k}}$$

$$c_p = \frac{\omega}{k} = \frac{\omega^2}{k\omega} = \underline{\frac{Dk^4 + S}{\rho k \omega}}$$

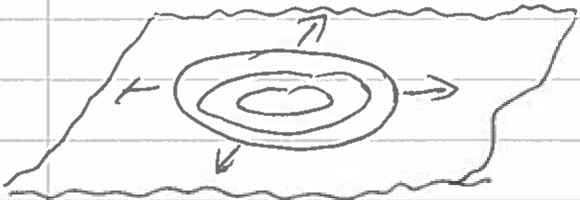
$$\Rightarrow \quad \underline{\underset{\sim}{c_g} / c_p = \frac{2Dk^3}{Dk^4 + S}\underset{\sim}{k}}$$

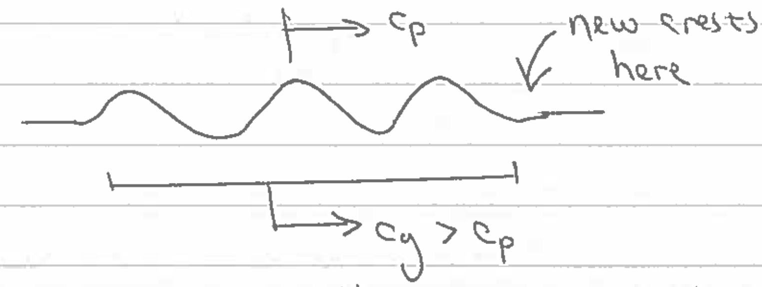
(b) cont.

(ii) ~~Let~~ let $k^* = (S/D)^{1/4} \Rightarrow \frac{c_g}{c_p} = \frac{2k^4}{k^4 + k^{*4}}$

For $\begin{cases} c_g > c_p \text{ need } & 2k^4 > k^4 + k^{*4} \Rightarrow k > k^* \\ c_g < c_p \text{ need } & 2k^4 < k^4 + k^{*4} \Rightarrow k < k^* \end{cases}$

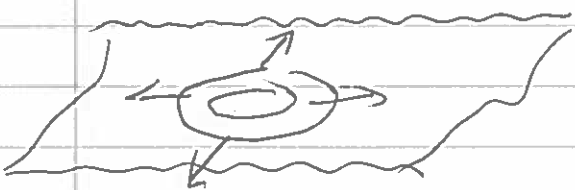
(iii) For $k > (S/D)^{1/4}$ we have $c_g > c_p$

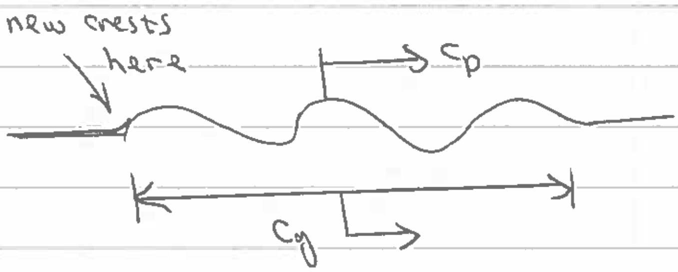




Since $c_g > c_p$ new crests continually appear at the right of the packet, ripple across the packet, and then dissapear at the left of the wave packet.

For $k < (S/D)^{1/4}$, we have $c_g < c_p$





This is like the deep water waves, with the crests appearing first on the left.

(2) (a) Diffusion length $l \sim \sqrt{\alpha t}$ is the distance heat can diffuse in a time t.

(b) Fist check it satisfies PDE.

$$\frac{\partial T}{\partial t} = -\frac{1}{2}\frac{T}{t} + \frac{x^2}{4\alpha t^2}T$$

$$\alpha\frac{\partial T}{\partial x} = \alpha\left(\frac{-2x}{4\alpha t}\right)T = -\frac{x}{2t}T$$

$$\Rightarrow \alpha\frac{\partial^2 T}{\partial x^2} = -\frac{1}{2t}T - \frac{x}{2t}\left(\frac{-2x}{4\alpha t}\right)T = -\frac{1}{2t}T + \frac{x^2}{4\alpha t^2}T$$

Thus, $\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$, as required.

Now check initial condition.

$$\lim_{t\to\infty}\frac{1}{\sqrt{t}}\exp\left[-\frac{x^2}{4\alpha t}\right] = \begin{cases}\frac{1}{\sqrt{t}} \to \infty & \text{for } x=0\\ 0 & \text{for } x\neq 0\end{cases}$$

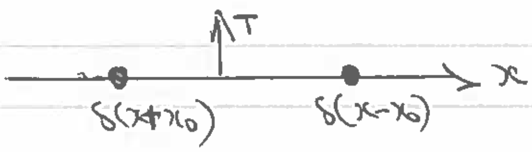
as required for δ function.

Also
$$\int_{-\infty}^{\infty} T\,dx = \frac{T_0}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp\left[-\frac{x^2}{4\alpha t}\right]\frac{dx}{2\sqrt{\alpha t}}$$

$$= \frac{T_0}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = T_0 \quad \checkmark$$

(c) The give solution corresponds to an initial condition in $-\infty < x < \infty$ of

$$T(x, t=0) = T_0\,\delta(x-x_0) + T_0\,\delta(x+x_0)$$



By symmetry, this satisfies $\frac{\partial T}{\partial x} = 0$ at $x=0$.

(d) By superposition, solution to $T(x,t=0) = T_0(x)$ is

$$T(x,t) = \int_0^\infty \frac{T_0(x')}{2\sqrt{\pi\alpha t}}\left[\exp\left(-\frac{(x-x')^2}{4\alpha t}\right) + \exp\left(-\frac{(x+x')^2}{4\alpha t}\right)\right] dx'$$

(the sum of the δ-function responces)

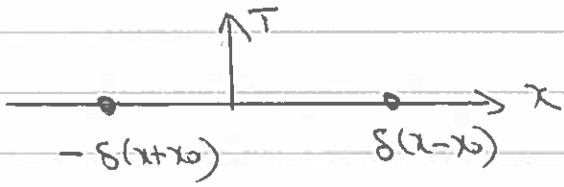
Check I.C. is satisfied:

For $t \to 0$ integrand is $T_0(x')\,\delta(x-x'_0)$, $0 < x < \infty$.

So $$\underset{(t\to 0)}{T(x,t)} = \int_{-\infty}^{\infty} T_0(x')\,\delta(x-x')\,dx' = T_0(x)$$ ✓

(e) For $-\infty < x < \infty$, an initial condition of

$$T(x,t=0) = T_0\,\delta(x-x_0) - T_0(x+x_0)$$



is antisymmetric in x, and so $T=0$ at $x=0$ for at time. Thus solution is

$$T(x,t) = \int_0^\infty \frac{T_0(x')}{2\sqrt{\pi\alpha t}}\left[\exp\left(-\frac{(x-x')^2}{4\alpha t}\right) - \exp\left(-\frac{(x+x')^2}{4\alpha t}\right)\right] dx'$$

(a) (i) $(\underline{e}\times\underline{x})\cdot(\underline{e}\times\underline{x}) = (\underline{e}\times\underline{x})_i\,(\underline{e}\times\underline{x})_i$

$$= \varepsilon_{ijk}\, e_j x_k\, \varepsilon_{ipq}\, e_p x_q = (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})$$

$$e_j e_p x_k x_q = e_p \cdot e_p x_q x_q - e_p e_q x_p x_q$$

$$= x_q x_q - e_p e_q x_p x_q \quad \text{as } |\underline{e}| = 1.$$

(ii)
$$\nabla\cdot\left[\frac{\underline{e}\times\underline{x}}{|\underline{e}\times\underline{x}|}\right] = \frac{\partial}{\partial x_i}\left[\varepsilon_{ijk} e_j x_k\,(x_q x_q - e_p e_q x_p x_q)^{-\frac{1}{2}}\right]$$

$$= \varepsilon_{ijk}\, e_j \delta_{ik} |\underline{e}\times\underline{x}|^{-1} - \frac{1}{2}\varepsilon_{ikj}\, e_j x_k\, |\underline{e}\times\underline{x}|^{-3/2}$$

$$(2x_q\delta_{iq} - e_p e_q \delta_{ip} x_q - e_p e_q x_p \delta_{iq})$$

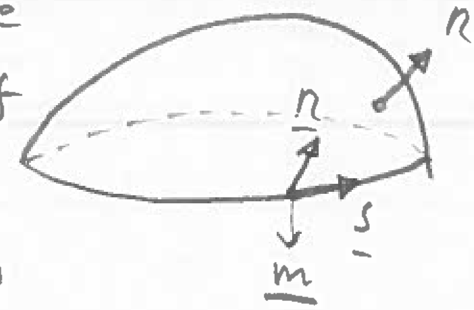
$$= \underbrace{\varepsilon_{ijk}\delta_{ik}}_{=0}\, e_j |\underline{e}\times\underline{x}|^{-1} - |\underline{e}\times\underline{x}|^{-3/2} e_j \underbrace{\varepsilon_{ijk} x_k x_i}_{=0}$$

$$+ \frac{1}{2}|\underline{e}\times\underline{x}|^{-3/2} x_k \underbrace{\varepsilon_{ijk} e_j e_i}_{=0} (e_q x_q + e_p x_p)$$

$= 0$ because $\varepsilon_{ijk}\delta_{ik}$, $\varepsilon_{ijk}x_k x_i$, $\varepsilon_{ijk}e_j e_i$ are symmetry / antisymmetry combinations.

(iii) If we choose $\underline{e}$ as the positive axial direction in a cylindrical coordinate system, $\frac{\underline{e}\times\underline{x}}{|\underline{e}\times\underline{x}|}$ is the unit vector in the azimuthal direction $\underline{e}_\theta$ outside the axial axis. It is obvious that it's divergence-free

(b) Let S is an orientable (possibly curved) surface with the unite vecter $\underline{n}$ normal to the surface S, $C = \partial S$ is the curve that bounds S, and $\underline{m}$ the unit vector normal of the surface S to the bond C, $\underline{s} = \underline{n} \times \underline{m}$



$d\underline{A} = \underline{n}\, dS$, $d\underline{l} = \underline{s}\, dc$, Let $\underline{f}$ a vector field

$$\iint_S (\nabla \times \underline{f}) \cdot d\underline{A} = \oint_C \underline{f} \cdot d\underline{l}$$

(c) Let $\underline{u} = (u(x,y), v(x,y))$ a 2D vector field in a 2D region S and C its bord,

$$\iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx\, dy = \oint_C (u\, dy - v\, dx)$$

Apply stokes to $\underline{f} = \begin{pmatrix} -v \\ u \\ 0 \end{pmatrix}$

$$\nabla \times \underline{f} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\iint_S (\nabla \times \underline{f}) \cdot d\underline{A} = \iint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx\, dy.$$

$$= \oint_C \begin{pmatrix} -v \\ u \\ 0 \end{pmatrix} \cdot d\underline{l} = \oint_C (-v\, dx + u\, dy).$$

(a) For small increments $\delta\theta$, $\delta\phi$, the distances moved on the surface of the sphere are $a\delta\theta$, $a\sin\theta\,\delta\phi$ respectively. These two movements are in orthogonal direction, so the total distance $\delta\ell$ satisfies

$$\delta\ell^2 = a^2\delta\theta^2 + a^2\sin^2\theta\,\delta\phi^2$$
$$= a^2\left[\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta\right]\delta\phi^2$$

$\therefore$ Total path length $L = a^2\int_{\phi_1}^{\phi_2}\left[(f')^2 + \sin^2 f\right]^{\frac{1}{2}} d\phi$

if $\theta = f(\phi)$.

a^2 is a constant, so need to minimise the given integral, $F = \left[(f')^2 + \sin^2 f\right]^{\frac{1}{2}}$.

(b) Euler-Lagrange Equation

$$\frac{\partial F}{\partial f} - \frac{d}{d\phi}\left(\frac{\partial F}{\partial f'}\right) = 0$$

$$= \frac{1}{\cancel{2}}\left[(f')^2 + \sin^2 f\right]^{\frac{1}{2}}(\cancel{2}\sin f\cos f) - \frac{d}{d\phi}\left[\frac{1}{\cancel{2}}\left[(f')^2+\sin^2 f\right]^{-\frac{1}{2}}\cancel{2}f'\right]$$

$$= \left[(f')^2+\sin^2 f\right]^{-\frac{1}{2}}\sin f\cos f + \frac{1}{2}\left[(f')^2+\sin^2 f\right]^{-3/2}(2f'f'' + 2\sin f\cos f\, f')f'$$
$$- f''\left[(f')^2 + \sin^2 f\right]^{-\frac{1}{2}}$$

$$\therefore \quad \sin f\cos f + \frac{(f'' + \sin f\cos f)(f')^2}{(f')^2 + \sin^2 f} - f'' = 0$$

(c) $f = \text{constant}$ means $f' = 0$, $f'' = 0$

$\therefore$ require $\sin f \cos f = 0$

So either $\sin f = 0$, ie $\theta = 0$ or π,
ie, a single point at the "N pole" or "S pole",

or $\cos f = 0$, ie $\theta = \pi/2$, ie., the path lies around the equator, which is a great circle.

(d) We are free to choose the axis about which we define our polar angles θ, ϕ. For any given pair of end points for the path, we can always choose our axis so that both points lie on the equator, keeping $\theta = \pi/2$. Could go either way - each is a local minimum, but the shorter of the two is the global minimum length (or geodesic)

Engineering Parts IIA and IIB 2018
4M12 – Partial Differential Equations and Variational Methods

Examiner's Report for IIB

Question 1: Group velocity applied to wave-like PDEs
Popular question with good performances from the students. Most marks were lost in part (b) in the drawing and interpretation of the dispersion pattern.

Question 2: Greens function solution of the diffusion equation
Another popular question, although the performance was not as good as for question 1. This is almost certainly because there has not been a similar question before. Marks were lost in the use of symmetry to meet boundary conditions.

Question 3: Index notation.
This involves elaborate calculation using index notation. Most students know the principle, but some were not able to carry the calculation out correctly to the end.

Question 4: Variational calculus.
This is an interesting question on the shortest path on a sphere, and the students showed good understanding of spherical coordinates and the variational principle. The easy way is to use the basic Euler-Lagrange equation, while some students used a particular form of this equation which makes the calculation less obvious.