Q1

(a) $\eta(x,t) = A(x,t)\, e^{j\theta(x,t)}$

Define $\underline{k = \partial\theta/\partial x}$, $\underline{\omega = -\partial\theta/\partial t}$

to recover local relationship

$$\eta = A\, e^{j(kx-\omega t)}$$

Thus $\frac{\partial k}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial t}\right) = -\frac{\partial\omega}{\partial x}$

Dispersion relationship is $\omega = \omega(k)$, so,

$$\frac{\partial k}{\partial t} = -\frac{d\omega}{dk}\frac{\partial k}{\partial x} = -c_g\frac{\partial k}{\partial x}$$

↑ defn. of c_g

$$\Rightarrow \quad \frac{\partial k}{\partial t} + c_g(k)\frac{\partial k}{\partial x} = 0 \qquad \text{——— ①}$$

Consider soln $k = f(x - c_g(k)t) = f(X)$

$$\frac{\partial k}{\partial t} = -f'(X)\frac{\partial}{\partial t}\left[c_g(k)t\right] = -f'(X)\left[c_g + c_g'(k)t\frac{\partial k}{\partial t}\right]$$

$$\Rightarrow \left[1 + f'(X)c_g'(k)t\right]\frac{\partial k}{\partial t} = -c_g f'(X) \qquad \text{——— ②}$$

Also, $\frac{\partial k}{\partial x} = +f'(X)\frac{\partial}{\partial x}\left[x - c_g(k)t\right] = +f'(X)\left[1 - c_g'(k)\frac{\partial k}{\partial x}t\right]$

$$\Rightarrow \left[1 + f'(X)c_g'(k)t\right]\frac{\partial k}{\partial x} = f'(X) \qquad \text{——— ③}$$

Equating ② and ③ shows $k = f(X)$ satisfies ①. So k is constant if observer moves such that $x = c_g t$.

(b) In 3D group velocity is a vector whose components are

$$[c_g]_i = \frac{\partial \omega}{\partial k_i}$$

3 Properties are: (i) $\underset{\sim}{c}_g$ is velocity at which wave packets propagate as a whole

(ii) $\underset{\sim}{c}_g$ is velocity at which to travel to keep seeing waves of wavevector $\underset{\sim}{k}$

(iii) $\underset{\sim}{c}_g$ is velocity at which the energy held in waves of wavevector $\underset{\sim}{k}$ propagates

(c) Trial solution $\underset{\sim}{u} = \underset{\sim}{u}_0 e^{j(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$, Then

$$\omega^2 k^2 \underset{\sim}{u}_0 e^{j(\wedge)} - (2\underset{\sim}{\Omega}\cdot\underset{\sim}{k})^2 \underset{\sim}{u}_0 e^{j(\wedge)}$$

$$\Rightarrow \quad \omega^2 = 4\frac{\Omega^2 k_z^2}{k^2} \quad \text{or} \quad \omega = \pm 2\Omega \frac{k_z}{k}$$

$$\begin{cases} \dfrac{\partial \omega}{\partial k_x} = \mp 2\Omega \dfrac{k_x k_z}{k^3} \\ \dfrac{\partial \omega}{\partial k_y} = \mp 2\Omega \dfrac{k_y k_z}{k^3} \\ \dfrac{\partial \omega}{\partial k_z} = \pm 2\Omega \dfrac{1}{k} \mp 2\Omega \dfrac{k_z^2}{k^3} = \pm 2\Omega \dfrac{k_x^2 + k_y^2}{k^3} \end{cases}$$

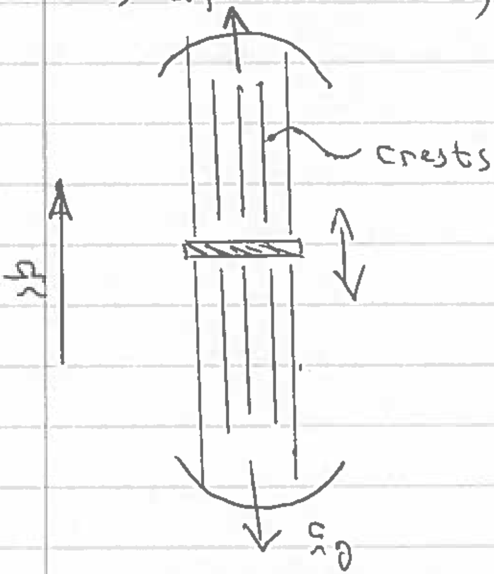
$$\underset{\sim}{c}_g = \pm \frac{2\Omega}{k^3}\left[-k_x k_z,\ -k_y k_z,\ k_x^2 + k_y^2\right]$$

compare $\underset{\sim}{c}_g = \pm \frac{2\Omega}{k^3}\left[\underset{\sim}{k}\times(\underset{\sim}{\hat{e}}_z \times \underset{\sim}{k})\right] = \pm \frac{2\Omega}{k^3}\left[k^2 \underset{\sim}{\hat{e}}_z - k_z \underset{\sim}{k}\right]$

$$= \pm \frac{2\Omega}{k^3}\left[-k_x k_z, -k_y k_z, k_x^2 + k_y^2\right]$$

Same ✓

(d) If $\omega \ll \Omega$, $k_z \simeq 0$, $\underset{\sim}{c}_g = \pm \frac{2\underset{\sim}{\Omega}}{k^3}(k_x^2 + k_y^2) = \pm \frac{2\underset{\sim}{\Omega}}{k}$



Both wave crests and $\underset{\sim}{c}_g$ are aligned with z-axis.

In general, $\underset{\sim}{c}_g$ is $\perp^{\text{lar}}$ to $\underset{\sim}{k}$, so $\underset{\sim}{c}_g$ is always aligned with wave crests

If $\omega = 2\Omega$, $k_x = k_y = 0$, $\underset{\sim}{c}_g = 0$

No wavepacket.

Examiners comments on Q1. Popular question mostly done very well. The primary difficulty was in part (a) where some students did not understand how to use the phase function to get the PDE for k. There were also some imaginative attempts at getting the dispersion pattern in part (d).

Q2

(a) $g(\underset{\sim}{x}) = \frac{1}{r} \Rightarrow \nabla^2 g = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} g$

$(r = |\underset{\sim}{x}|)$ $\qquad = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = 0$

$(|\underset{\sim}{x}| \neq 0)$

$\nabla g = -\frac{\underset{\sim}{x}}{r^3} \Rightarrow \oint \nabla g \cdot d\underset{\sim}{s} = -\oint \frac{ds}{R^2} = -4\pi$.

(R is radius of surface)

Now solution of $\nabla^2 g = -\delta$ must satisfy:

(i) $\nabla^2 g = 0$ for $\underset{\sim}{x} \neq 0$

(ii) $\int \nabla^2 g \, dV = \oint \nabla g \cdot d\underset{\sim}{s} = -\int \delta \, dV = -1$

Clearly $g = \frac{1}{4\pi r}$ satisfies both of these.

(b) Use superposition. If δ-function sits at $\underset{\sim}{x}'$ and has strength s' then from (a)

$$g = \frac{s'}{4\pi|\underset{\sim}{x} - \underset{\sim}{x}'|}$$

Approximate continuous distribution $s(\underset{\sim}{x})$ by a set of δ-functions distributed in space. The superposition gives

$$g(\underset{\sim}{x}) = \frac{1}{4\pi}\int \frac{s(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} dV'$$

(c) $\nabla^2 \underset{\sim}{B} = -\mu_0 \nabla \times \underset{\sim}{J} = -\underset{\sim}{S}(\underset{\sim}{x})$, $\underset{\sim}{S} = \mu_0 \nabla \times \underset{\sim}{J}$

Each component of this inverts as in part(b), so

$$\underset{\sim}{B} = \frac{1}{4\pi}\int \frac{\underset{\sim}{S}(\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,dV' = \frac{\mu_0}{4\pi}\int \frac{[\nabla \times \underset{\sim}{J}]'}{|\underset{\sim}{r}|}\,dV'$$

$$\Rightarrow \quad \underset{\sim}{B} = \frac{\mu_0}{4\pi}\int \frac{\nabla' \times \underset{\sim}{J}(\underset{\sim}{x}')}{|\underset{\sim}{r}|}\,dV'$$

(d) $\nabla' \times \left(\underset{\sim}{J}'/|\underset{\sim}{r}|\right) = \nabla'\left(\frac{1}{|\underset{\sim}{r}|}\right) \times \underset{\sim}{J}' + \frac{1}{|\underset{\sim}{r}|}\nabla' \times \underset{\sim}{J}'$

But $\nabla'\left(\frac{1}{|\underset{\sim}{r}|}\right) = +\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}$, so

$$\nabla' \times \left(\underset{\sim}{J}'/|\underset{\sim}{r}|\right) = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3} \times \underset{\sim}{J}' + \frac{1}{|\underset{\sim}{r}|}\nabla' \times \underset{\sim}{J}'$$

c.f. $\nabla\left(\frac{1}{|\underset{\sim}{r}|}\right) = -\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}$, $\nabla'\left(\frac{1}{|\underset{\sim}{r}|}\right) = -\frac{(-\underset{\sim}{r})}{|\underset{\sim}{r}|^3} = +\frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3}$

Thus $\frac{\nabla' \times \underset{\sim}{J}'}{|\underset{\sim}{r}|} = \nabla' \times \left(\frac{\underset{\sim}{J}'}{|\underset{\sim}{r}|}\right) - \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|^3} \times \underset{\sim}{J}'$

and

$$\underset{\sim}{B} = \frac{\mu_0}{4\pi}\int \nabla' \times \left(\frac{\underset{\sim}{J}'}{|\underset{\sim}{r}|}\right) dV' - \frac{\mu_0}{4\pi}\int \frac{\underset{\sim}{r} \times \underset{\sim}{J}'}{|\underset{\sim}{r}|^3}\,dV'$$

↑ converts to a surface integral at infinity

If current localised in space

$$\underset{\sim}{B} = -\frac{\mu_0}{4\pi}\int \frac{\underset{\sim}{r} \times \underset{\sim}{J}'}{|\underset{\sim}{r}|^3}\,dV'$$

Examiners comments on Q2. Popular question mostly done well. The weakest part was part (b) where the manner in which superposition needs to be used to construct the Green's function was often rather vague.

3 A uniform inextensible chain, of length l, and mass m, hangs under gravity between two fixed pegs which are a horizontal distance $L < l$ apart.

(a) If the chain is parameterized by an intrinsic arc-length coordinate $-l/2 < s < l/2$, show that the total gravitational potential energy of the chain is

$$V = \frac{mg}{l}\int_{-l/2}^{l/2}\left(\int_{-l/2}^{s}\sin(\theta(s'))\mathrm{d}s'\right)\mathrm{d}s,$$

where $\theta(s)$ is the angle the chain at s makes with the horizontal.

The chain has mass per unit length $\rho = m/l$.

Using GPE=mgh, the potential for an increment of chain ds is:

$$\mathrm{d}V = \rho \mathrm{d}s g y(s) = \frac{mg}{l}y(s)\mathrm{d}s.$$

Integrating along the chain gives the total potential energy

$$V = \frac{mg}{l}\int_{0}^{l} y(s)\mathrm{d}s.$$

For the same increment of chain ds the change in y is

$$\mathrm{d}y = \mathrm{d}s\sin\theta.$$

so the total height of the chain at s is

$$y(s) = \int_{-l/2}^{s}\sin(\theta(s'))\mathrm{d}s'.$$

Putting these two results together, the total gravitational potential energy is

$$v = \frac{mg}{l}\int_{-l/2}^{l/2} y(s)\mathrm{d}s = \frac{mg}{l}\int_{-l/2}^{l/2}\left(\int_{-l/2}^{s}\sin(\theta(s'))\mathrm{d}s'\right)\mathrm{d}s$$

(b) Find two integral constraints on the function $\theta(s)$, which ensure the chain hangs between the pegs? [10%]

The chain needs to arrive at the second peg, so we need

$$x(l/2) = x(-l/2) + L$$

and

$$y(l/2) = y(-l/2).$$

In terms of $\theta(s)$, the second of these constraints is

$$\int_{-l/2}^{l/2}\sin(\theta(s))\mathrm{d}s = 0$$

and, by analogy, the first is

$$\int_{-l/2}^{l/2} \cos(\theta(s))\mathrm{d}s = L.$$

(c) By minimizing the constrained potential energy, show that the chain adopts the form

$$\theta = \tan^{-1}(As)$$

and explain what sets the value of A. You do not need to actually find the value of A.

Using Lagrange multipliers, we need to minimize

$$V = \frac{mg}{l}\int\left(\int_{-l/2}^{s}\sin(\theta(s'))\mathrm{d}s'\right)\mathrm{d}s + \lambda_1\int_{-l/2}^{l/2}\sin(\theta(s))\mathrm{d}s + \lambda_2\left(\int_{-l/2}^{l/2}\cos(\theta(s))\mathrm{d}s - L\right).$$

$$V = \int_{-l/2}^{l/2}\left[\frac{mg}{l}\left(\int_{-l/2}^{s}\sin(\theta(s'))\mathrm{d}s'\right) + \lambda_1\sin(\theta(s)) + \lambda_2\left(\cos(\theta(s)) - \frac{L}{l}\right)\right]\mathrm{d}s.$$

Integrating the first term by parts, we get

$$V = \int_{-l2/}^{l/2} -\frac{mg}{l}s\sin(\theta(s)) + \lambda_1\sin(\theta(s)) + \lambda_2\left(\cos(\theta(s)) - \frac{L}{l}\right)\mathrm{d}s.$$

Minimizing this function with respect to $\theta(s)$, we get

$$0 = -\frac{mg}{l}s\cos(\theta(s)) + \lambda_1\cos(\theta(s)) - \lambda_2\sin(\theta(s)),$$

which we solve for $\theta(s)$, to get

$$\tan\theta = \frac{\lambda_1 - \frac{mgs}{l}}{\lambda_2}.$$

On symmetry grounds, we know $\theta(0) = 0$, giving $\lambda_1 = 0$, so we have

$$\tan\theta = -\frac{mgs}{\lambda_2 l} \equiv As.$$

To find A we need to find λ_2, which we would do from by implementing the x constraint

$$\int_{-l/2}^{l/2} \cos(\theta(s))\mathrm{d}s = L.$$

(d) A second inextensible chain, which also has length l, and mass total m, is hung between the pegs. However, this chain has a non-uniform mass per unit length $\rho(s)$. What function, $\rho(s)$, is required for the chain to hang in the arc of a circle with radius R.

The potential energy for a varying density chain is

$$V = \int_{-l/2}^{l/2}\left[g\rho(s)\left(\int_{-l/2}^{s}\sin(\theta(s'))\mathrm{d}s'\right) + \lambda_1\sin(\theta(s)) + \lambda_2\left(\cos(\theta(s)) - \frac{L}{l}\right)\right]\mathrm{d}s.$$

Now when we integrate by parts we get:

$$V = \int_{-l/2}^{l/2} -g\left[\int_{-l/2}^{s} \rho(s')\mathrm{d}s'\right]\sin(\theta(s)) + \lambda_1 \sin(\theta(s)) + \lambda_2\left(\cos(\theta(s)) - \frac{L}{l}\right)\mathrm{d}s,$$

and when we minimize on $\theta(s)$ we get:

$$0 = -g\left[\int_{-l/2}^{s} \rho(s')\mathrm{d}s'\right]\cos(\theta(s)) + \lambda_1 \cos(\theta(s)) - \lambda_2 \sin(\theta(s)),$$

For the arc of a circle of radius R we need the energy to be minimized by

$$\theta(s) = \frac{s}{R}.$$

Substituting and rearranging, we get

$$g\int_{-l/2}^{s} \rho(s')\mathrm{d}s' = \lambda_1 - \lambda_2 \tan(s/R).$$

Differentiating with respect to s gives the form of the density:

$$\rho(s) = -\frac{\lambda_2}{gR}\sec^2(s/R).$$

To fix λ_2*, we need the total mass of the chain to be m*

$$\int_{-l/2}^{l/2} \rho(s)\mathrm{d}s = -\frac{2\lambda_2}{g}\tan\left(\frac{l}{2R}\right) = m$$

so the final answer is:

$$\rho(s) = \frac{m}{2R}\cot\left(\frac{l}{2R}\right)\sec^2\left(\frac{s}{R}\right).$$

Examiners comments on Q3. An unpopular question with a low average. Most candidates did well at (a) and (b). Part (c) was poorly answered, with very few candidates applying integration by parts. Not a single candidate made a serious attempt on (d), and the mark scheme was adjusted to bring most of these marks forward to earlier parts of the question.

4 (a) Evaluate the following quantities, which are written using (three dimensional) index notation.

$$\delta_{ij}\delta_{ij}$$

This is one whenever i and j are the same, and zero otherwise, so it sums to three.

$$\delta_{ij}\delta_{ij} = 3$$

$$\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{ijk}\varepsilon_{lmn}$$

This is equivalent to

$$(\varepsilon_{ijk}\varepsilon_{ijk})^2$$

$\varepsilon_{ijk}\varepsilon_{ijk}$ *is one when ijk are cyclic (three possibilities) or anticyclic (three possibilities) so sums to 6.*

Therefore

$$\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon_{ijk}\varepsilon_{lmn} = 36$$

$$\varepsilon_{ijk}\varepsilon_{rjk}$$

Here i and r are free indices. If they are equal, the object is one for ijk=rjk cyclic (once) or anti-cyclic (once).

If they are not equal, one of the two epsilons is always zero, because if $i \neq j \neq k$ *then j or k must be equal to r.*

Hence

$$\varepsilon_{ijk}\varepsilon_{rjk} = 2\delta_{ir}$$

(b) (i) Compute, from first principles, $DI(y^{(n)})[z]$, the directional derivative of the functional given by:

$$I = \int_0^1 \left(\frac{d^n y}{dx^n}\right)^2 \mathrm{d}x.$$

By the definition of the directional derivative

$$DI(y^{(n)})[z] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\int_0^1 \left(\frac{d^n y}{dx^n} + \varepsilon\frac{d^n z}{dx^n}\right)^2 \mathrm{d}x - \int_0^1 \left(\frac{d^n y}{dx^n}\right)^2 \mathrm{d}x\right)$$

$$DI(y^{(n)})[z] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_0^1 2\frac{d^n y}{dx^n}\varepsilon\frac{d^n z}{dx^n} + \mathscr{O}(\varepsilon^2)\mathrm{d}x$$

$$DI(y^{(n)})[z] = \int_0^1 2\frac{d^n y}{dx^n}\frac{d^n z}{dx^n}\mathrm{d}x$$

(ii) Find the differential equation and boundary conditions that $y(x)$ must satisfy to minimize I. [20%]

We now need to integrate by parts n times. First doing it once

$$DI(y^{(n)})[z] = \int_0^1 -2\frac{d^{n+1}y}{dx^{n+1}}\frac{d^{n-1}z}{dx^{n-1}}\mathrm{d}x + 2\frac{d^n y}{dx^n}\frac{d^{n-1}z}{dx^{n-1}}\bigg|_0^1$$

then doing it $n-1$ *more times, we get*

$$DI(y^{(n)})[z] = \int_0^1 2\,(-1)^n\,\frac{d^{2n}y}{dx^{2n}}z\;\mathrm{d}x + \sum_{i=0}^{n-1} 2\frac{\mathrm{d}^{n+i}y}{\mathrm{d}x^{n+i}}\;\frac{\mathrm{d}^{n-1-i}z}{\mathrm{d}x^{n-1-i}}\bigg|_0^1$$

Setting the directional derivative of this integral to zero, the governing equation is

$$\frac{d^{2n}y}{dx^{2n}} = 0$$

and, from the boundary terms, we see the boundary conditions will be

$$\frac{d^i y}{dx^i}\bigg|_0 = 0 \qquad \frac{d^i y}{dx^i}\bigg|_1 = 0$$

for $i = n, n+1, n+2.....2n-1$

(ii) Find the set of functions $y(x)$ that minimize I subject to the constraint

$$\int_0^1 y(x)\mathrm{d}x = 1,$$

and find the minimum value of I.

We use a Lagrange multiplier to implement the constraint, considering the new functional

$$I + \lambda\left(\int_0^1 y(x)\mathrm{d}x - 1\right)$$

Setting the directional derivative of this integral to zero, we now need

$$2\,(-1)^n\,\frac{d^{2n}y}{dx^{2n}} + \lambda = 0$$

which, integrating 2n times, is solved by

$$y = \sum_{i=0}^{2n} A_i x^i$$

where A_i are constants of integration, except A_{2n} which is a constant from the particular integral proportional to λ.
The boundary terms from the directional derivative have not changed, so we still need

$$\left.\frac{d^i y}{dx^i}\right|_0 = 0 \qquad \left.\frac{d^i y}{dx^i}\right|_1 = 0$$

for $i = n, n+1, n+2.....2n-1$
The x=0 boundary conditions require $A_i = 0$ for $i = n, n+1...2n-1$.
The x=1 boundary conditions are then all satisfied provided $A_{2n} = 0$.
The minimizing set of functions is thus

$$y = \sum_{i=0}^{n-1} A_i x^i$$

Finally we use the constraint to fix A_0, giving the final form for the set of functions as

$$y(x) = 1 - \sum_{i=1}^{n-1} \frac{A_i}{i+1} + \sum_{i=1}^{n-1} A_i x^i,$$

where the constants A_i can take any value.
This makes good sense because, with this set of functions, the value of I is zero.

Examiners comments on Q4. Popular question mostly done well. Many candidates scored full marks on the index notation section. Many candidates correctly identified the differential equation in (b)(ii) but surprisingly few also identified the correct boundary conditions, often instead imposing clamped/Dirichlet conditions without justification. Complete answer’s to (b)(iii) were rare.