Part IB, Paper 1, 2016, Solutions

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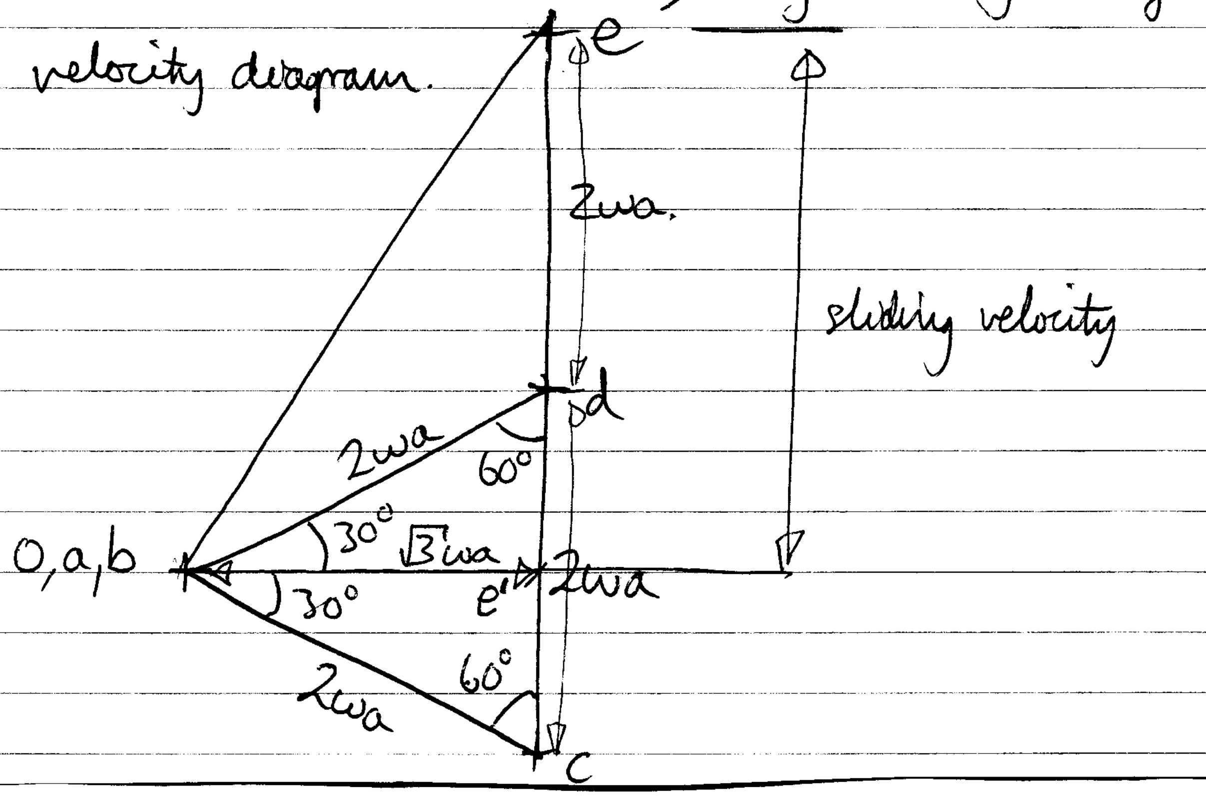
1 a) velocity diagram.



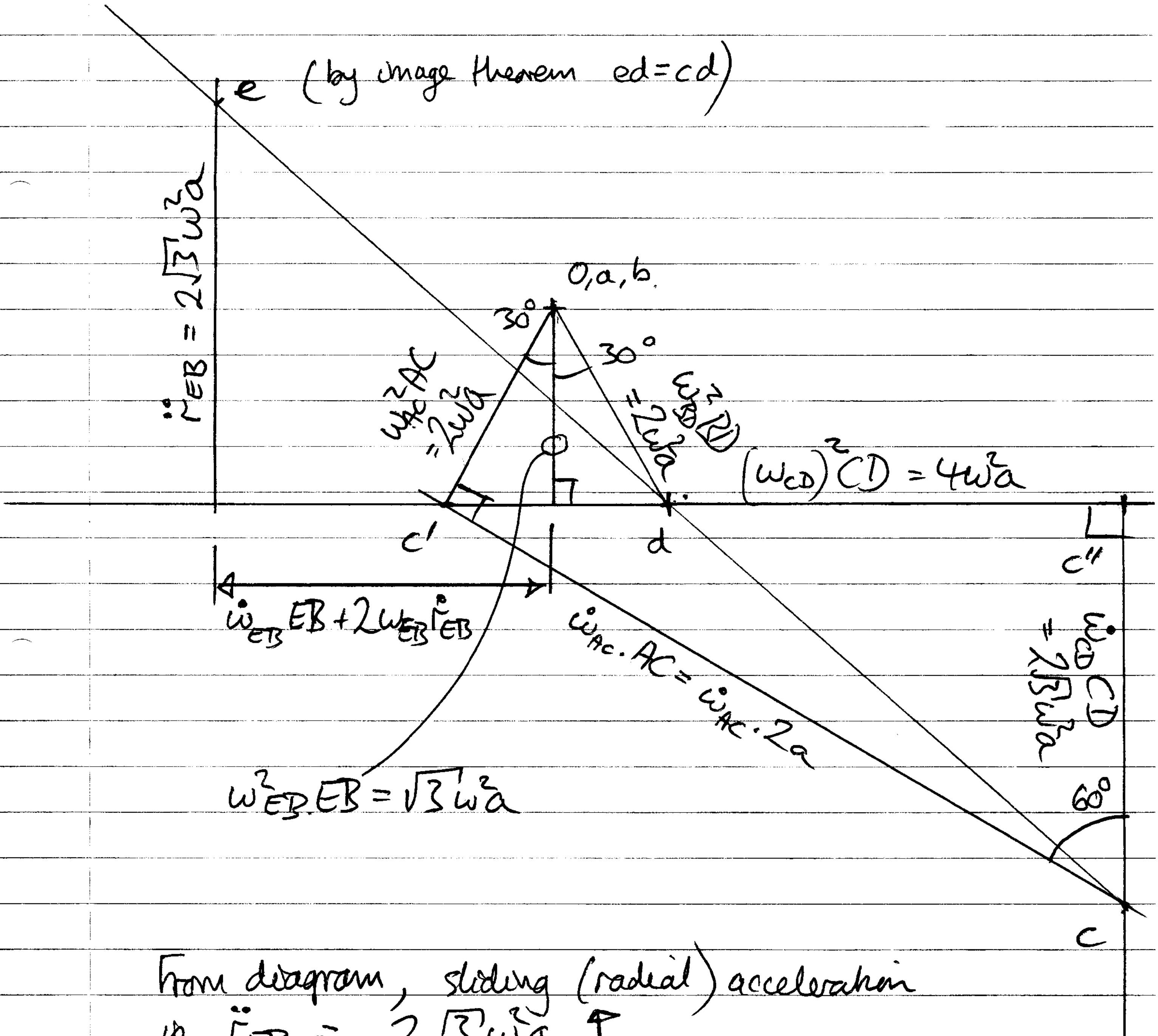
sliding velocity = 3ωa (extending)

$$\omega_{CDE} = \frac{ce}{CE} = \frac{4\omega a}{2a} = 2\omega$$

$$\omega_{AC} = \frac{ac}{AC} = \frac{2\omega a}{2a} = \omega$$

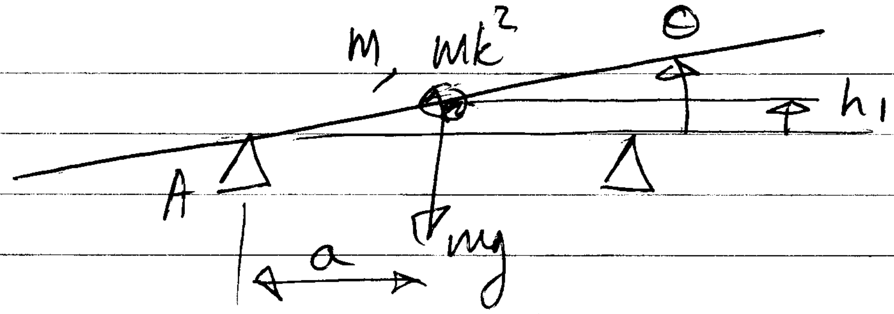
$$\omega_{BE} = \frac{be'}{BE} = \frac{\sqrt{3}\,\omega a}{\sqrt{3}\,a} = \omega$$

b) acceleration diagram.



From diagram, sliding (radial) acceleration is $\ddot{r}_{EB} = 2\sqrt{3}\,\omega^2 a \uparrow$

2 a)



moments about A

$$mga = -(mk^2 + ma^2)\ddot{\theta}$$

$$\ddot{\theta} = -\frac{mga}{mk^2 + ma^2} = -\frac{ga}{k^2+a^2}$$

integrate $\quad \dot{\theta} = -\frac{gat}{k^2+a^2} + C$

$\dot{\theta} = 0$ when $t = 0 \quad \therefore \; C = 0$

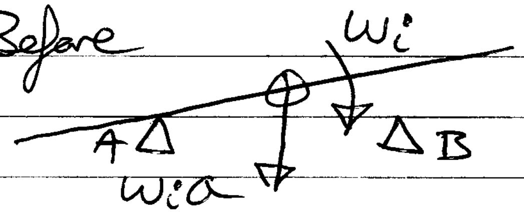
integrate $\quad \theta = -\frac{gat^2}{2(k^2+a^2)} + D$

$\theta = \frac{h_1}{a}$ at $t = 0 \quad \therefore \; D = \frac{h_1}{a}$

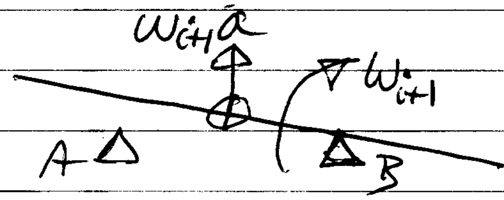
$$\theta = -\frac{gat^2}{2(k^2+a^2)} + \frac{h_1}{a}$$

time to $\theta = 0 \qquad T_1 = \sqrt{\frac{2h_1(k^2+a^2)}{ga^2}}$

b) Before



After



Impact at B, so conservation of moment of momentum about B

$$mk^2\omega_i - m\omega_i a^2 = mk^2\omega_{i+1} + m\omega_{i+1}a^2$$

$$\frac{\omega_{i+1}}{\omega_i} = \frac{k^2-a^2}{k^2+a^2}$$

conservation of energy: PE lost = KE gained

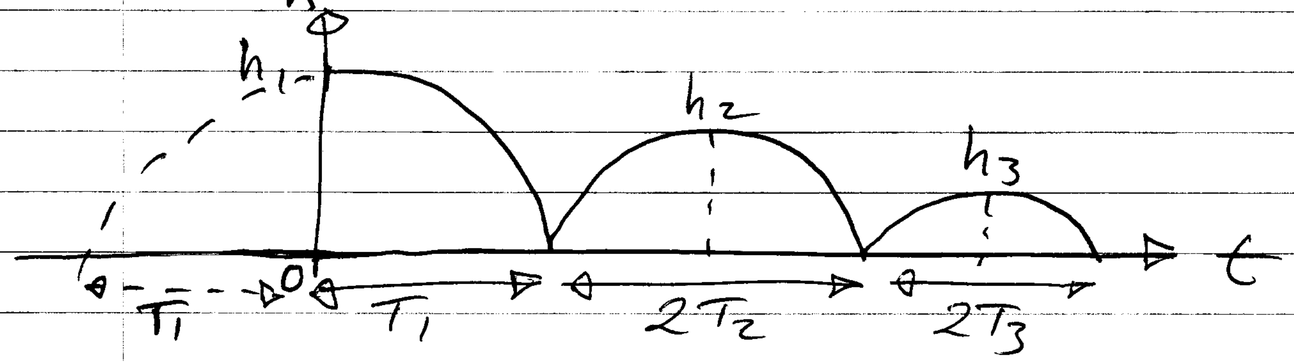
$$mgh_i = \frac{1}{2}\omega_i^2(mk^2 + ma^2)$$

where h_i is max. height before impact i
ω_i is ang. velocity immediately before impact i

$$\therefore\ h_i \propto \omega_i^2, \quad \frac{h_{i+1}}{h_i} = \frac{\omega_{i+1}^2}{\omega_i^2} = \underline{\underline{\left(\frac{k^2-a^2}{k^2+a^2}\right)^2}}$$

c) $k = 2a \therefore T_1 = \sqrt{\dfrac{2h_1(4a^2+a^2)}{ga^2}} = \sqrt{h_1}\sqrt{\dfrac{10}{g}}$

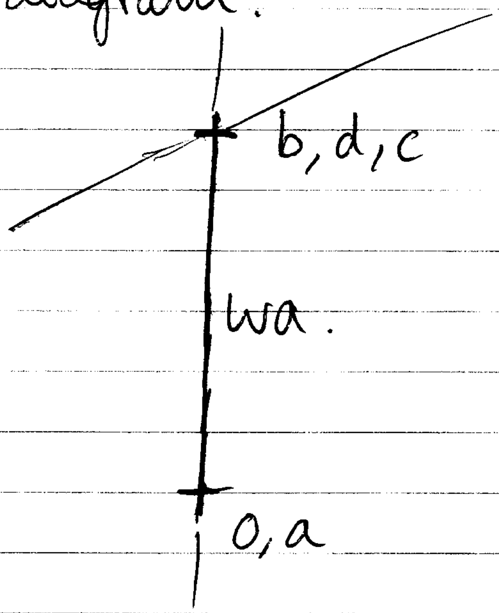
and $\dfrac{h_{i+1}}{h_i} = \left(\dfrac{4a^2-a^2}{4a^2+a^2}\right)^2 = \left(\dfrac{3}{5}\right)^2 \quad \therefore \sqrt{\dfrac{h_{i+1}}{h_i}} = \dfrac{3}{5}$



$$\begin{aligned}
t_\infty &= 2T_1 + 2T_2 + 2T_3 + \dots - T_1 \\
&= 2(T_1 + T_2 + T_3 + \dots) - T_1 \\
&= 2\left(\sqrt{h_1}\sqrt{\frac{10}{g}} + \sqrt{h_2}\sqrt{\frac{10}{g}} + \sqrt{h_3}\sqrt{\frac{10}{g}} + \dots\right) - \sqrt{h_1}\sqrt{\frac{10}{g}} \\
&= 2\sqrt{h_1}\sqrt{\frac{10}{g}}\left(1 + \sqrt{\frac{h_2}{h_1}} + \sqrt{\frac{h_2}{h_1}}\sqrt{\frac{h_3}{h_2}} + \dots - \frac{1}{2}\right) \\
&= 2\sqrt{h_1}\sqrt{\frac{10}{g}}\left(\underbrace{1 + \frac{3}{5} + \left(\frac{3}{5}\right)^2 + \dots}_{GP,\ S_\infty = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}} - \frac{1}{2}\right)
\end{aligned}$$

$$\therefore\ t_\infty = \underline{\underline{4\sqrt{\frac{10h_1}{g}}}}$$

3. a) velocity diagram.

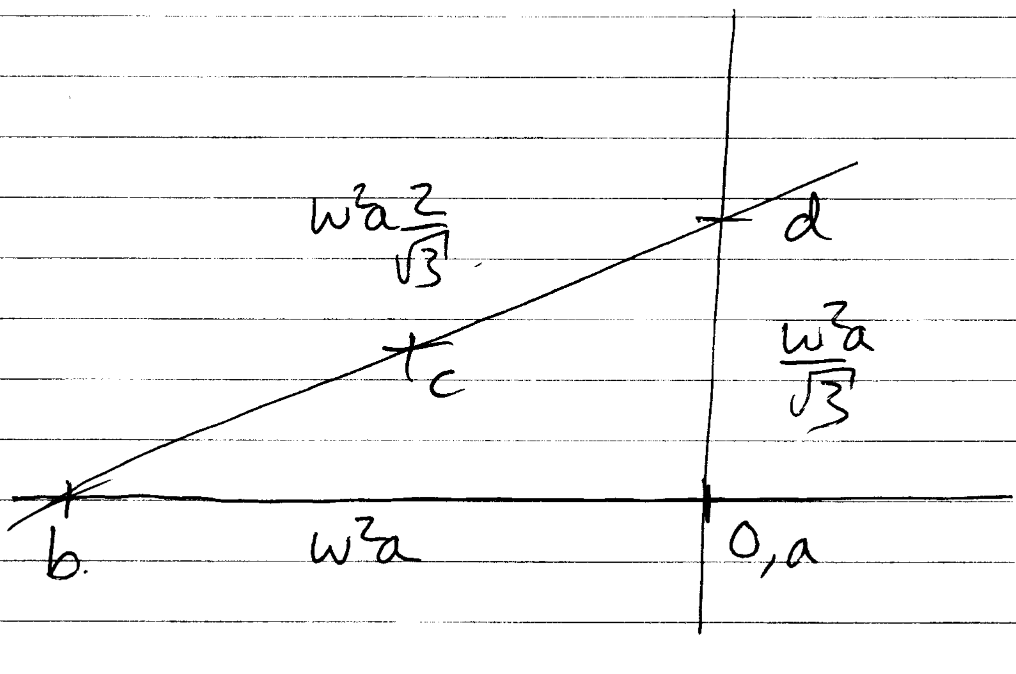


velocity of piston D $\underline{V_D} = \omega a \uparrow$

angular velocity of rod BD $\underline{\omega_{BD}} = 0.$

velocity of rod at C $\underline{V_C} = \omega a \uparrow.$

b) acceleration diagram.



From (a) the angular velocity of the rod is zero, So no centripetal acceleration component between B and D.

acceleration of piston D $\underline{a}_D = \frac{\omega^2 a}{\sqrt{3}} \uparrow$

angular accn of rod BD $\underline{\dot{\omega}}_{BD} = \frac{bd}{BD} = \frac{\omega^2 a \frac{2}{\sqrt{3}}}{2a}$

$= \frac{\omega^2}{\sqrt{3}} \circlearrowright$

accn of rod at C $\underline{a}_C = \begin{cases} \frac{\omega^2 a}{2\sqrt{3}} \uparrow \\ \frac{\omega^2 a}{2} \leftarrow \end{cases}$

c) power.

$\underline{Q} \cdot \underline{\omega} = M_P \underline{a}_D \cdot \underline{V}_D + M_R \underline{a}_C \cdot \underline{V}_C + I \underline{\dot{\omega}}_{BD} \cdot \underline{\omega}_{BD} + \underline{F} \cdot \underline{V}_D$

$= M_P \cdot \frac{\omega^2 a}{\sqrt{3}} \cdot \omega a + M_R \frac{\omega^2 a}{2\sqrt{3}} \omega a + M_R \frac{\omega^2 a}{2} \cdot 0$

$+ I \frac{\omega^2}{\sqrt{3}} 0 + F . \omega a$

$\therefore Q = M_P \frac{\omega^2 a^2}{\sqrt{3}} + M_R \frac{\omega^2 a^2}{2\sqrt{3}} + Fa$ $\|$

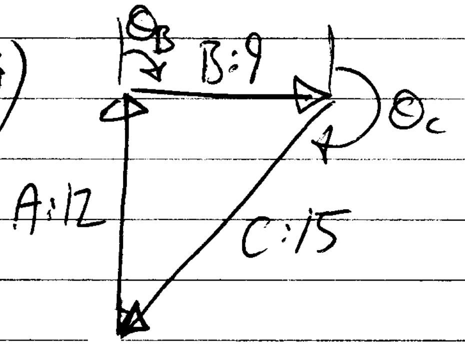
applied to the crank
in same direction as rotation ω

4 a) static balance – the resultant of all the bearing forces is zero.

dynamic balance – the total moment of the bearing reaction forces about any point on the shaft is zero.

For three unbalanced rotors, static balance can be achieved the the sum of the two smaller out of balances is greater than or equal to the largest out of balance.

b) i)

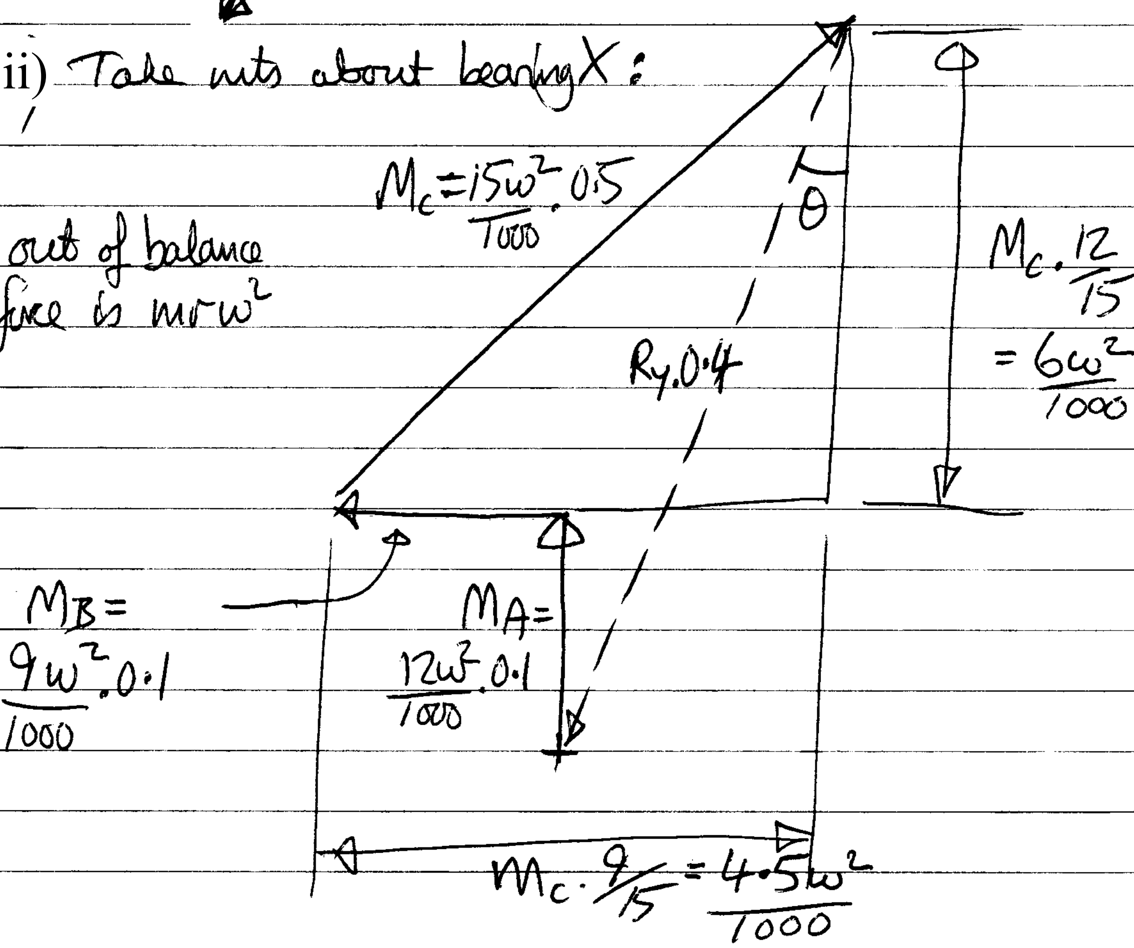


$\Theta_B = 90^\circ$

$\Theta_C = 180^\circ + \sin^{-1}\frac{9}{15} = 216.9^\circ$

ii) Take mts about bearing X:

out of balance force is $mr\omega^2$



out of balance moment

$$R_y . 0.4 = \sqrt{\left(M_C \frac{12}{15} + M_A\right)^2 + \left(M_C \frac{9}{15} - M_B\right)^2}$$

$$= \sqrt{\left(\frac{6\omega^2}{1000} + \frac{1.2\omega^2}{1000}\right)^2 + \left(\frac{4.5\omega^2}{1000} - \frac{0.9\omega^2}{1000}\right)^2}$$

$$= \frac{\omega^2}{1000} \sqrt{7.2^2 + 3.6^2} = \frac{\omega^2}{1000} \cdot \frac{18}{\sqrt{5}}$$

$$R_y \, 0.4 = \left(5000 . \frac{2\pi}{60}\right)^2 \frac{1}{1000} \frac{18}{\sqrt{5}} = \underline{\underline{2207 \text{ Nm}}}$$

$$R_y = \underline{\underline{5517 \text{N}}}$$ (Rx has same magnitude but opposite direction).

$$\theta = \tan^{-1} \frac{3.6}{7.2} = \underline{\underline{26.6^\circ}}$$

iii) remove mass m at radius r at A and C to generate moment

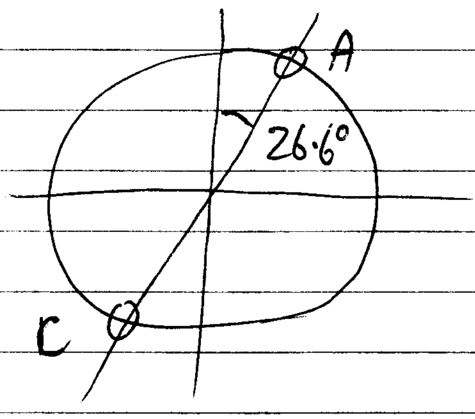
$$m r \omega^2 0.6 = 2207 \text{ Nm}$$

r = 0.1m.   distance between A and C

$$\text{mass } m = \frac{2207}{0.6 . 0.1 \left(\frac{5000 . 2\pi}{60}\right)^2}$$

$$\underline{\underline{m = 0.134 \text{ kg}}}$$

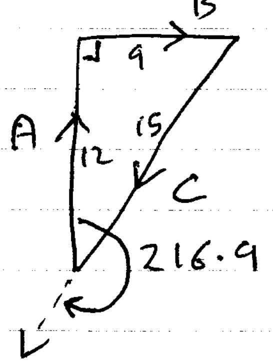
removed at



Alternative solution to Q4

4 a) Static balance: C of G on shaft axis, no *net* bearing forces
Dynamic balance: bearing forces both zero
Can achieve static balance if no one imbalance is greater than the sum of the other two.

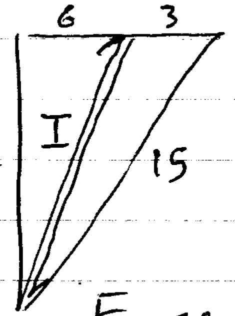
b) i)



Rotate B by 90°, and C by 216.9°.

(3-4-5 triangle)

ii) Move $\frac{2}{3}$ of B → A, $\frac{1}{3}$ of B → C:

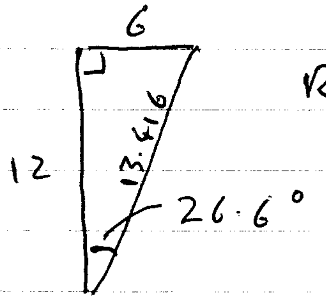


$$I = \sqrt{12^2 + 6^2} = 13.416$$

$$\omega = \frac{10000\pi}{60} = 523.6 \text{ rad/s}$$

$$\text{Force} = \frac{13.416 \times 0.6\,\omega^2}{0.4 \times 1000} \times 10^{-3} = \underline{5.517 \text{ kN}}$$

iii)



Remove 13.416 Kg mm @ 0.1 m
i.e. 0.134 Kg @ 26.6° on A and same amount in opposite direction on C.

5 a) no slipping between gears so:

$$\omega_1 N_A = \omega_2 N_B \quad \text{and} \quad \omega_2 N_C = \omega_3 N_D$$

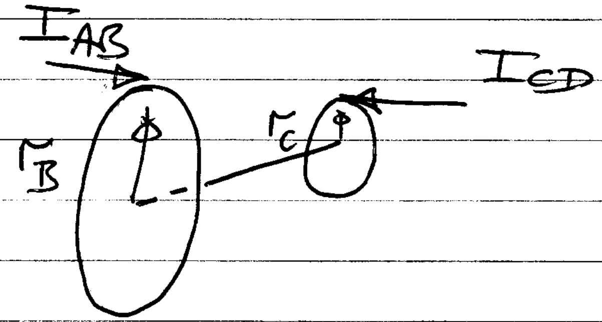
where N_i are the number of teeth.

$$\therefore \quad \frac{\omega_1}{\omega_2} = \frac{N_B}{N_A} \quad \text{and} \quad \frac{\omega_2}{\omega_3} = \frac{N_D}{N_C}$$

$$\therefore \quad \frac{\omega_1}{\omega_3} = \frac{\omega_1}{\omega_2} \cdot \frac{\omega_2}{\omega_3} = \frac{N_B}{N_A} \cdot \frac{N_D}{N_C} = \frac{36}{12} \cdot \frac{30}{18} = \underline{\underline{5}}$$

b) gear contact forces generate moments on the shafts of the rotors.

c)



equilibrium of shaft BC about its axis
no shaft inertia $\therefore$

$$I_{AB} \cdot r_B = I_{CD} \cdot r_C$$

$$\frac{I_{CD}}{I_{AB}} = \frac{r_B}{r_C} = \frac{N_B}{N_C} = \frac{36}{18} = \underline{\underline{2}}$$

Note $N_A + N_B = N_C + N_D$
therefore radius of gear wheels is proportional to number of teeth.

d) angular momentum before and after:

	BEFORE	AFTER
J :	ωJ	$\omega' J$
$2J$:	$\omega 2J$	$\frac{\omega'}{5} 2J$

change in momentum arises from impulse at gear contact * radius of gear wheel:

J: $\omega J + I_{AB} . 12 = \omega' J$ —— (1)

$2J$: $\omega 2J - I_{CD} . 30 = \frac{\omega'}{5} . 2J$

from (c) $I_{CD} = 2 I_{AB}$

$\therefore \omega 2J - 2 I_{AB} 30 = \frac{\omega'}{5} . 2J$ —— (2)

(1) $\times 5$: $\omega 5J + I_{AB} 60 = 5 \omega' J$

(1) $\times 5$ + (2) : $\omega 7J = \frac{27}{5} \omega' J$

$$\omega' = \frac{35}{27} \omega$$

e) KE before $= \frac{1}{2}(J + 2J)\omega^2 = \frac{1}{2} J \omega^2 . 3$

KE after $= \frac{1}{2} J \omega'^2 + \frac{1}{2} 2J \left(\frac{\omega'}{5}\right)^2$

$= \frac{1}{2} J \omega'^2 \left(1 + \frac{2}{5^2}\right)$

$= \frac{1}{2} J \omega^2 \left(\frac{35}{27}\right)^2 . \frac{27}{25}$

Hence fraction of KE lost is $\dfrac{3 - \left(\frac{35}{27}\right)^2 . \frac{27}{25}}{3}$

$= 1 - \frac{7.7}{3.27} = 1 - \frac{49}{81} = \frac{32}{81}$

Alternative solution to Q5

5 a) $\dfrac{\omega_1}{\omega_3} = \dfrac{n_B}{n_A} \times \dfrac{n_D}{n_C} = \dfrac{36}{12} \times \dfrac{30}{18} = \underline{5}$

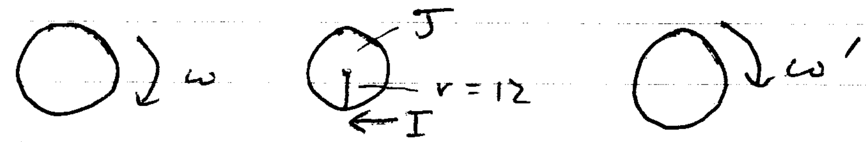
b) Because not all external impulses pass through (or are parallel to) the centre lines of shafts 1 and 4.

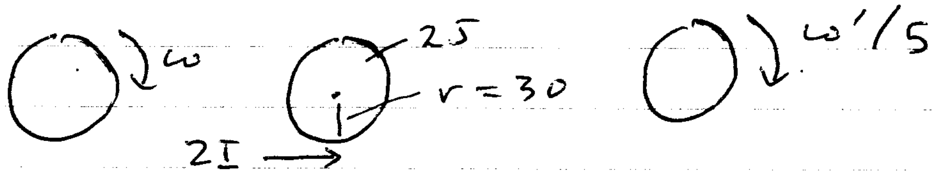
c) $\dfrac{n_B}{n_C} = \dfrac{36}{18} = 2$

ALSO, $n_A + n_B = n_C + n_D$, so all teeth have the same pitch, and $r \propto n$

$\therefore \dfrac{r_B}{r_C} = 2$, and $J = 0$ for shaft 2

So $I_{CD} = 2 \times I_{AB}$

d) 1: 

4: 

So: $J(\omega' - \omega) = 12\,I$ (1)

$2J(\omega - \omega'/5) = 60\,I$ (2)

$\omega'(5 + 2/5) = \omega(2 + 5)$

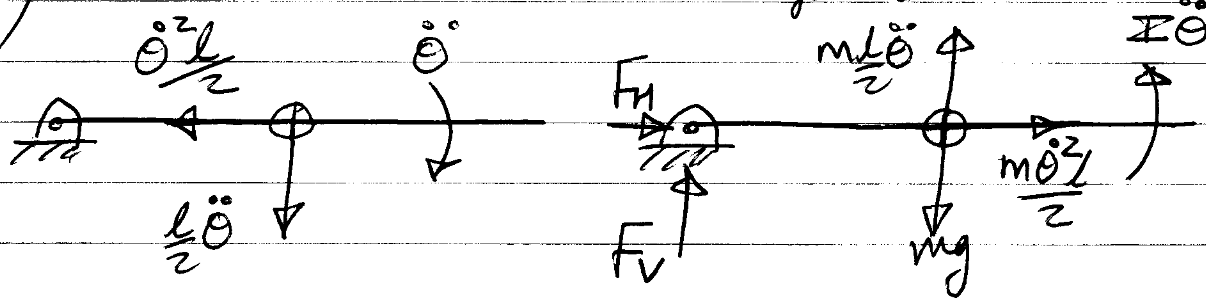
$\omega' = \omega \times \dfrac{7 \times 5}{27} = \underline{1.296\,\omega}$

e) $E_1 = \tfrac{3}{2} J \omega^2$ $E_2 = \tfrac{J}{2}\omega'^2 + J\left(\tfrac{\omega'}{5}\right)^2$

$\dfrac{E_2}{E_1} = \dfrac{54 \times 2}{100 \times 3}\left(\dfrac{\omega'}{\omega}\right)^2 = 0.605$

$\therefore$ Fraction Lost $= \underline{0.395}$

6 a) accns. forces



mts about pivot

$$mg\frac{l}{2} - \frac{ml}{2}\ddot\theta\frac{l}{2} - I\ddot\theta = 0 \qquad I = \frac{ml^2}{12}$$

$$\frac{mgl}{2} = \ddot\theta\left(\frac{ml^2}{12} + \frac{ml^2}{4}\right)$$

$$\ddot\theta = \frac{mgl/2}{ml^2/3} = \underline{\underline{\frac{3}{2}\frac{g}{l}}}$$

b) energy PE lost = KE gained

$$mg\frac{l}{2} = \frac{1}{2}I\dot\theta^2 + \frac{1}{2}m\left(\frac{\dot\theta l}{2}\right)^2$$

$$= \frac{1}{2}ml^2\left(\frac{1}{12} + \frac{1}{4}\right)\dot\theta^2$$

$$\underline{\underline{\dot\theta^2 = \frac{3g}{l}, \quad \dot\theta = \sqrt{\frac{3g}{l}}}}$$

c) reactions.

sum forces vertically $\uparrow$

$$F_V - mg + ml\ddot\theta/2 = 0$$

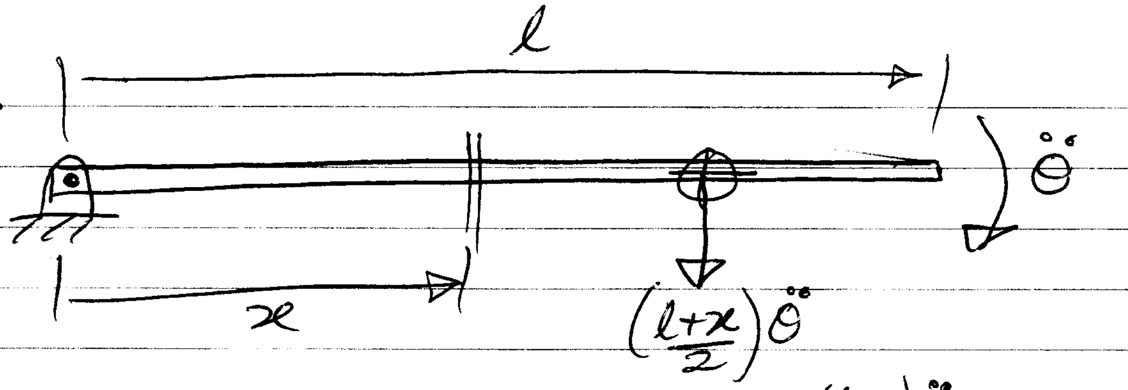
$$F_V = mg - ml\frac{3}{2}\frac{g}{l}\cdot\frac{1}{2} = mg\left(1 - \frac{3}{4}\right) = \underline{\underline{\frac{mg}{4}}}$$

sum forces horizontally $\rightarrow$

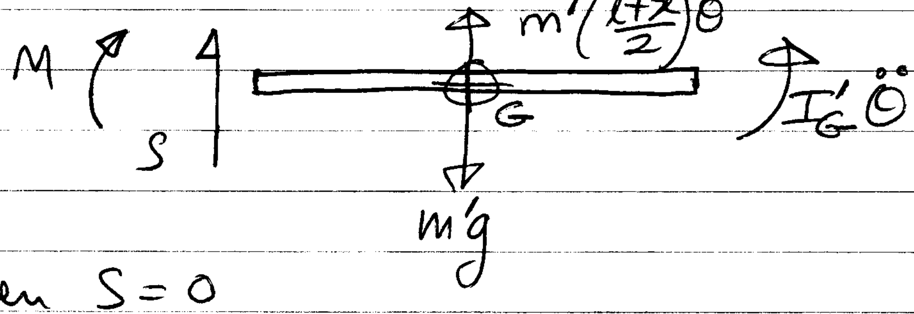
$$F_H + m\dot\theta^2 l/2 = 0$$

$$F_H = -\frac{ml}{2}\left(\frac{3g}{l}\right) = \underline{\underline{-\frac{3}{2}mg}}$$

d) accns



forces



max BM when $S = 0$
sum forces vertically

$$\cancel{m'}g = \cancel{m'}\left(\frac{l+x}{2}\right)\ddot{\theta}$$

$$\cancel{g} = \left(\frac{l+x}{2}\right)\frac{3}{2}\frac{\cancel{g}}{l}\cancel{\sin 90} \to 1$$

$$4l = 3l + 3x.$$

$$x = \frac{l}{3}$$

mts about centre of cut beam, mass m', length $(l-x)$

$$M = I'_G\,\ddot{\theta}$$

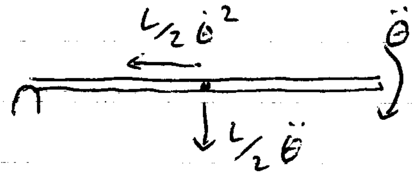
$$= \frac{1}{12}m'(l-x)^2 \cdot \frac{3}{2}\frac{g}{\cancel{l}}\sin 90$$

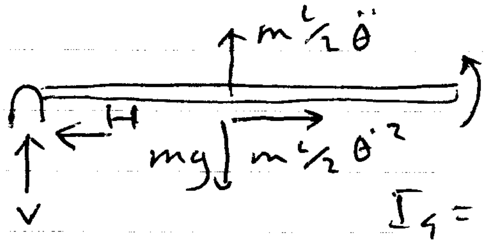
$$= \frac{1}{12}\cdot\frac{2m}{\cancel{3}}\left(\frac{2}{3}l\right)^2\frac{\cancel{3}}{\cancel{2}}\frac{g}{l}$$

$$M = \frac{4mgl}{12\cdot 9} = \frac{mgl}{27}$$

Alternative solution to Q6

6 a)





$I_G = \frac{mL^2}{12}$

T.M.A. pivot,

$$mg\frac{L}{2} = m\left(\frac{L^2}{4} + \frac{L^2}{12}\right)\ddot\theta \rightarrow \ddot\theta = \frac{3g}{2L}$$

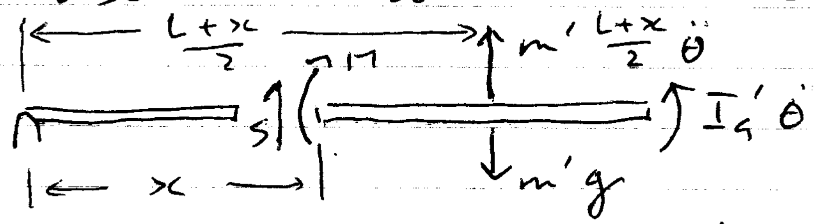
b) PE lost $= \frac{mgL}{2}$, KE $= \frac{m}{2}\left(\frac{L\dot\theta}{2}\right)^2 + \frac{mL^2}{24}\dot\theta^2$

$$\frac{g}{2} = \frac{L\dot\theta^2}{6} \rightarrow \dot\theta = \sqrt{\frac{3g}{L}}$$

c) R.V., $V = m\left(g - \frac{L}{2}\ddot\theta\right) = \frac{mg}{4} \uparrow$

R.H., $H = m\frac{L}{2}\dot\theta^2 = \frac{3mg}{2} \leftarrow$

d) Max BM where SF = 0



$S = 0$ when $g = \frac{L+x}{2}\ddot\theta = \frac{3g(L+x)}{2L} \rightarrow x = \frac{L}{3}$

Now $m' = \frac{2m}{3}$, $I_G' = \frac{2m}{3}\frac{4L^2}{9\times 12} = \frac{2mL^2}{81}$

$$\therefore M = I_G'\ddot\theta = \frac{mgL}{27}$$

Examiners' comments

Question 1: planar kinematics (four-bar mechanism and slider)

Part (a) was answered well by most candidates. Recurring problems were: incorrect or missing direction information; diagrams too small (recommended scale was ignored); image theorem used but points CDE in wrong order; sliding velocity and angular velocity of BE calculated using total velocity of E rather than parallel and perpendicular components.

Part (b) was also answered well. Recurring problems were: centripetal acceleration neglected when calculating radial acceleration of E; centripetal acceleration along CD drawn in opposite direction. A small minority of solutions involved: assigning unit vectors; writing a position vector expression; and differentiating twice to determine acceleration components. Unfortunately, nearly all attempts using this approach were unsuccessful, due to inappropriate choice of unit vectors, incorrect position vector expression, or errors in differentiation. Solutions that involved tabulating the four acceleration components in each link were much more successful.

Question 2: energy and momentum (rocking beam)

This was the least popular question in section A. In part (a) most solutions involved either using an energy expression to find the velocity just before impact or Newton’s 2nd Law to find acceleration, then SUVAT to find time. A significant number of solutions assumed that the centre of mass accelerated downwards at 9.8 m s^{-2}, ignoring any effect of the moment of inertia. Occasionally dimension *a* was confused with acceleration.

In part (b) most solutions recognised that moment of momentum was conserved about the point of impact (at B), but there was often difficulty in writing a correct expression for the moment of momentum before impact (wrong signs).

Part (c) proved to be quite difficult although many solutions recognised the need for a geometric progression and a sum to infinity. Some fruitless attempts involved summing energy lost in impacts and equating to the initial potential energy.

Question 3: planar kinematics and virtual power (crank and piston)

Part (a) involved finding velocities. Although a velocity diagram was not specifically requested (unlike Q1) almost all solutions included a diagram. A common error was to place point c incorrectly (midway between a and d instead of b and d). The term ‘translational velocity’ was sometimes misunderstood (‘left-right velocity’ rather than the intended ‘not angular velocity’). Some solutions appeared to disregard the possibility that BD could have zero angular velocity.

The acceleration diagram in part (b) sometimes incorrectly included a non-zero centripetal acceleration term for BD, even though the angular velocity was correctly found in part (a) to be zero.

Most solutions to part (c) involved a virtual power equation, but a variety of errors meant few completely correct answers. Common problems were: neglect of direction information when finding the scalar product of force and velocity; inclusion of gravity (plan view); neglect of subscripts in the two mass parameters. A few solutions found the answer quickly by drawing a free body diagram of the piston and rod and considering equilibrium of forces in the AD direction.

Answers were often stated without direction information, or with units when there are none given in the question. Marking of this question was sometimes made difficult by untidy presentation, especially answers buried in the middle of the page and almost indistinguishable from the rest of the working.

Question 4: balancing of rotating shaft

Generally competently answered. Only a handful of candidates gave a full answer to part (a), but almost all were able to answer part (b) without difficulty (though several drew a figure or applied the cosine rule before realising that they were dealing with a 3:4:5 triangle). Most who solved part (c) did so by taking moments in two planes, though a few spotted how the central rotor could be 'split' between the outer ones, to simplify the problem.

Question 5: angular momentum

This question was not popular, and proved difficult for most who attempted it. Many candidates tried to answer part (c) by equating the ratio of the two impulses to the ratio of the two inertias; and of those who related it to the gear radii on shaft 2, only two even mentioned the (necessary, and provable) assumption that all the gears must have the same pitch. Only a handful of candidates approached part (d) in a sensible way, and just one was able to find the final angular velocities in the gearbox correctly.

Question 6: planar kinematics and bending moments

This question was answered by the vast majority of candidates. Nearly all candidates could answer parts (a) – (c) perfectly. Almost all candidates drew the bar in a 'general' position (as in the figure in the question) and a substantial majority then spent extra effort finding general expressions for angular velocity and acceleration, before evaluating these at the requested position. Several candidates clearly knew in advance that the maximum bending moment should be at the L/3 position, and checked/adjusted their algebra accordingly.

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