EGT1
ENGINEERING TRIPOS PART IB

Thursday 2 June 2016 2 to 4

Paper 6

INFORMATION ENGINEERING: SOLUTIONS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper, graph paper, semilog graph paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

*Answer not more than **two** questions from this section.*

1 (a) (i) The transfer function from u to y is given by the ratio $G(s) = \bar{y}(s)/\bar{u}(s)$ for $\dot{y}(0) = y(0) = 0$. Take the Laplace transform of the dynamical equation:

$$m(s^2\bar{y}(s) - sy(0) - \dot{y}(0)) + c(s\bar{y}(s) - y(0)) = \bar{u}(s)$$

Noting that $\dot{y}(0) = y(0) = 0$, we can rearrange to give:

$$(ms^2 + cs)\bar{y}(s) = \bar{u}(s)$$

which then gives us the transfer function as

$$G(s) = \frac{1}{(ms^2 + cs)} = \frac{1}{s(ms + c)} = \frac{\frac{1}{m}}{s(s + \frac{c}{m})}$$

[3]

[*Many students forgot the role of initial conditions in the Laplace transform.*]

(ii) Now let us expand the transfer function $G(s)$ as partial fractions:

$$G(s) = \frac{\frac{1}{m}}{s(s + \frac{c}{m})} = \frac{\alpha}{s} + \frac{\beta}{s + \frac{c}{m}}$$

From the expression above, we can write

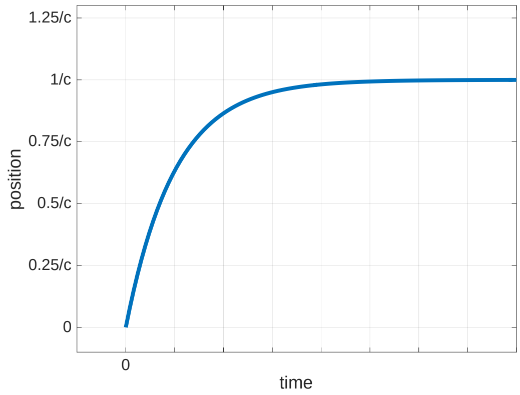
$$G(s) = \frac{(\alpha + \beta)s + \alpha c/m}{s(s + \frac{c}{m})}$$

We therefore have $(\alpha + \beta) = 0$ and $\alpha c/m = 1/m$, giving $\alpha = 1/c$ and $\beta = -1/c$.

$$G(s) = \frac{1}{c}\left(\frac{1}{s} - \frac{1}{s + c/m}\right)$$

We can now invert this to give (using databook identities for LTs):

$$g(t) = \mathscr{L}^{-1}(G(s)) = \alpha\mathscr{L}^{-1}\left(\frac{1}{s}\right) + \beta\mathscr{L}^{-1}\left(\frac{1}{s + \frac{c}{m}}\right) = \frac{1}{c}\left(1 - e^{-\frac{c}{m}t}\right) \text{ for } t \geq 0\,.$$



[7]

(iii) $g(t) = \frac{1}{c}\left(1 - e^{-\frac{c}{m}t}\right)$.

For each $t \geq 0$ we have that $0 \leq 1 - e^{-\frac{c}{m}t} \leq 1$ therefore $|g(t)| = g(t) \leq \frac{1}{c}$. Thus,

$$\int_0^T |g(t)|\,dt \;=\; \int_0^T g(t)\,dt \;\leq\; \int_0^T \frac{1}{c}\,dt \;\leq\; \frac{T}{c}\,.$$

It follows that any $A > 0$ and $B = \frac{1}{c}$ guarantee $\int_0^T |g(t)|dt < A + BT$. [5]

[*The reasoning here was sometimes confused: a number of students did not find a constant positive value for the bound A but defined A as a function of time T.*]

(b) (i) The poles of $T(s)$ are roots of the polynomial $s^2 + (1+k_p)s + k_i = 0$:
If $k_i = (1+k_p)^2/4$, this polynomial becomes

$$s^2 + (1+k_p)s + \frac{(1+k_p)^2}{4} = \left[s + \frac{1}{2}(1+k_p)\right]^2$$

therefore we have roots/poles at $s = -\frac{1}{2}(1+k_p)$. [3]

(ii) For $k_p = 0$ the transfer function reads $T(s) = \frac{k_i}{s^2+s+k_i}$
The poles occur at $s = (1/2)(-1 \pm \sqrt{1-4k_i})$, so that we have complex roots if $k_i > 1/4$ as given. Write the denominator as

$$s^2 + s + k_i = s^2 + 2\xi\omega_n s + \omega_n^2$$

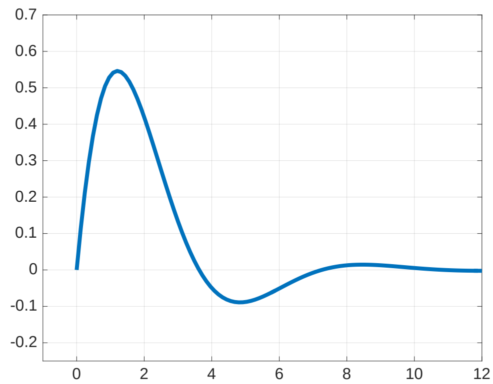
Thus, $\omega_n^2 = k_i$ and $2\xi\omega_n = 1$, that is,

$$\omega_n = \sqrt{k_i} \qquad\qquad \xi = \frac{1}{2\sqrt{k_i}}$$

For a damping factor of $\xi = 0.5$, we see that $k_i = 1$ and therefore $\omega_n = 1$.
For $k_i = 1$ we have a denominator of $s^2 + s + 1 = (s + 1/2)^2 + 3/4$. Recall that a function of the form $e^{-\sigma t} \sin \omega t$ has a LT of $\frac{\omega}{(s+\sigma)^2+\omega^2}$. Therefore our $T(s)$ which is of the form $\frac{1}{(s+1/2)^2+3/4}$ has an impulse response proportional to

$$e^{-\sigma t} \sin(\omega t)$$

where $\sigma = 0.5$ and $\omega = \sqrt{0.75} = \frac{\sqrt{3}}{2}$. The impulse response is drawn below. [7]



[*Some candidates sketched the impulse response with a non-zero final bias; oscillations and exponential decay were neglected.*]

2 (a) (i) For $D = 0$, we have $G(s) = 1/2s$. Therefore $G(j\omega) = \frac{1}{2j\omega} = \frac{1}{2\omega} e^{-j\pi/2}$. Giving

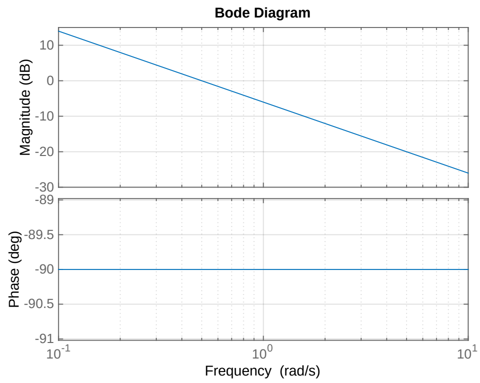
$$|G(j\omega)| = \frac{1}{2\omega} \quad \angle G(j\omega) = -\frac{\pi}{2}$$

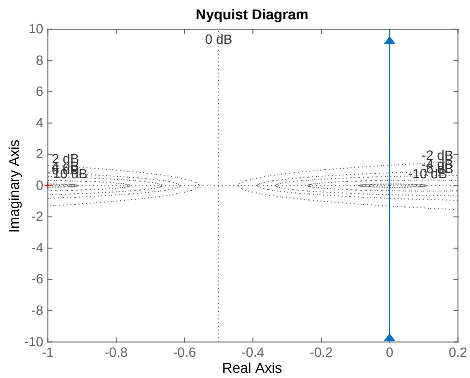
Thus on the Bode diagram, the phase (semi-log plot of phase v frequency) is simply a horizontal line at $-\pi/2$, as shown below.
The Bode diagram for magnitude is a log-log plot of frequency vs magnitude (in dB): $20 \log |G(j\omega)| = -20 \log(2\omega) = -20 \log 2 - 20 \log \omega$. Thus we have a constant slope of $-20\text{dB}/\text{dec}$ (integrator), passing through $-20 \log(2) =\simeq -6\text{dB}$ at $\omega = 1$. This is also shown below.

The Nyquist diagram plots the real vs imaginary parts of $G(j\omega)$. Since $G(j\omega) = -j/(2\omega)$ has no real part, the plot will be along the imaginary axis (i.e. constant

 (cont.

phase of $-\pi/2$). From the Bode diagram, or just taking $-1/(2\omega)$ as $\omega \to 0$ from above and below: we see that this line starts at $\lim\limits_{\omega \to 0^+} = -\infty$, passes through 0 since $\lim\limits_{\omega \to \infty} |G(j\omega)| = 0$ then continues toward $\lim\limits_{\omega \to 0^-} = \infty$.
The Nyquist diagram is shown below.

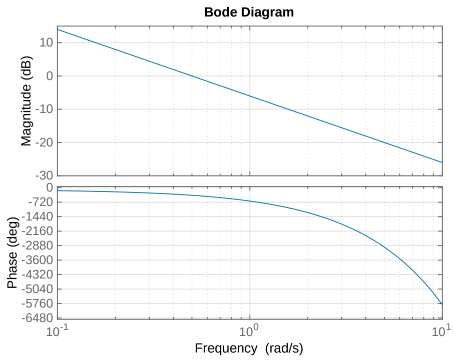


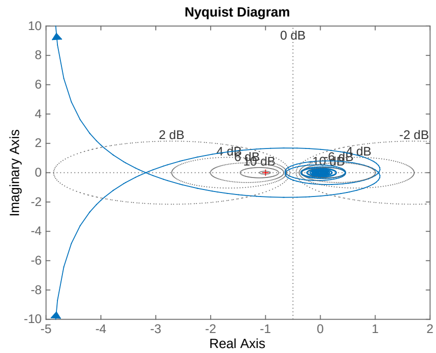


Gain margin is ∞ (phase never reaches $-\pi$, so no intersection with the negative real axis). [5]

[*Many candidates did not sketch the Nyquist locus for* $\omega \in (-\infty, 0)$.]

(ii) A delay introduces a phase shift of $-j\omega D$. The Nyquist locus will therefore tend to the vertical asymptote crossing the real axis at $-kD$ as $\omega \simeq 0$ since $G(j\omega) \simeq k\frac{1-j\omega D}{j\omega} = \frac{k}{j\omega D} - kD$, while wrapping around the origin for $\omega \to \infty$. The gain margin reduces as D increases. Above a certain delay threshold the system becomes unstable. Bode and Nyquist diagrams for $D = 10$ and $k = \frac{1}{2}$ are reported below. [4]





(iii) Input and output have same amplitude when at crossover frequency $\omega_c = 0.5$ since

$$|G(j\omega_c)| = \left|\frac{e^{-D\omega_c j}}{2\omega_c j}\right| = \frac{1}{2\omega_c} = 1\ .$$

[3]

[*A few students did not notice that delays do not affect the amplitude of the transfer function.*]

(iv) Constant unit input: $\bar{u}(s) = \frac{1}{s}$. Output: $\bar{y}(s) = G(s)\bar{u}(s) = \frac{1}{2s^2}$. Antitransform: $y(t) = \frac{1}{2}t$. The output increases with t and is therefore unbounded. [3]

[*Most of the answers to this part mentioned the unboundedness of the Bode diagram for $s \to 0$ instead of exploiting the direct argument based on the antitransform.*]

(b) (i)

$$\bar{y}_c(s) = K(s)(\bar{r}(s) - \bar{y}(s))$$

and

$$\bar{y} = G(s)\bar{u}(s) = G(s)[\bar{y}_c(s) + \bar{d}(s)]$$

Therefore: $\bar{y}(s) = G(s)[\bar{y}_c(s) + \bar{d}(s)]$ and

$$\bar{y}(s) = G(s)K(s)(\bar{r}(s) - \bar{y}(s)) + G(s)\bar{d}(s)$$

Giving

$$\bar{y}(s) = \frac{G(s)K(s)}{1+G(s)K(s)}\bar{r}(s) + \frac{G(s)}{1+G(s)K(s)}\bar{d}(s)$$

Therefore, our transfer functions are: $\frac{G(s)K(s)}{1+G(s)K(s)}$ between $\bar{r}$ a [5]

(ii) The transfer function from $\bar{r}(s)$ to $\bar{y}(s)$ is given by

$$W_{r,y}(s) = \frac{\frac{k_p}{2s}}{1+\frac{k_p}{2s}} = \frac{\frac{k_p}{2s}}{\frac{2s+k_p}{2s}} = \frac{k_p}{k_p+2s} \quad \Rightarrow \quad y(t) = W_{r,y}(0) = 1$$

at steady state. The transfer function from $\bar{d}(s)$ to $\bar{y}(s)$ is given by

$$W_{d,y}(s) = \frac{\frac{1}{2s}}{1+\frac{k_p}{2s}} = \frac{\frac{1}{2s}}{\frac{2s+k_p}{2s}} = \frac{1}{k_p+2s} \quad \Rightarrow \quad y(t) = W_{d,y}(0) = \frac{1}{k_p}$$

at steady state. By linearity, the steady state output is $y(t) = W_{r,y}(0) + W_{d,y}(0) = 1 + \frac{1}{k_p}$. [5]

[*Some students wasted time here by computing the complete output response instead of the steady state response.*]

3 (a) Laplace transforms:

$$s\bar{h}_1 = \bar{q}_0 - \bar{q}_1 \qquad s\bar{h}_2 = \bar{q}_1 - \bar{q}_2 \qquad \bar{q}_1 = \alpha(\bar{h}_1 - \bar{h}_2) \qquad \bar{q}_2 = \beta\bar{h}_2 \,.$$

Assuming $h_1(0) = h_2(0) = 0$.
Solve for $\bar{h}_1$:

$$s\bar{h}_1 = \bar{q}_0 - \alpha(\bar{h}_1 - \bar{h}_2) \quad \Rightarrow \quad (s+\alpha)\bar{h}_1 = \bar{q}_0 + \alpha\bar{h}_2 \quad \Rightarrow \quad \bar{h}_1 = \frac{\bar{q}_0 + \alpha\bar{h}_2}{(s+\alpha)}$$

Solve for $\bar{h}_2$:

$$\begin{aligned} s\bar{h}_2 &= \alpha(\bar{h}_1 - \bar{h}_2) - \beta\bar{h}_2 = \alpha\left(\frac{\bar{q}_0 + \alpha\bar{h}_2}{(s+\alpha)} - \bar{h}_2\right) - \beta\bar{h}_2 = \\ &= \alpha\frac{\bar{q}_0 + \alpha\bar{h}_2 - (s+\alpha)\bar{h}_2}{(s+\alpha)} - \beta\bar{h}_2 = \alpha\frac{\bar{q}_0 - s\bar{h}_2}{(s+\alpha)} - \beta\bar{h}_2 \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad \left(s + \frac{\alpha s}{s+\alpha} + \beta\right)\bar{h}_2 = \frac{\alpha}{(s+\alpha)}\bar{q}_0 \\ &\Rightarrow \quad \frac{s^2 + (2\alpha+\beta)s + \alpha\beta}{s+\alpha}\bar{h}_2 = \frac{\alpha}{(s+\alpha)}\bar{q}_0 \\ &\Rightarrow \quad \bar{h}_2 = \frac{\alpha}{s^2 + (2\alpha+\beta)s + \alpha\beta}\bar{q}_0 \,. \end{aligned}$$

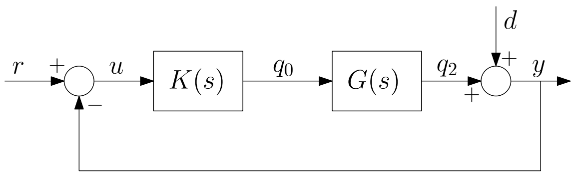
Finally,

$$\frac{\bar{q}_2(s)}{\bar{q}_0(s)} = \frac{\alpha\beta}{s^2 + (2\alpha+\beta)s + \alpha\beta} \,.$$

[9]

[*Most students were able to complete this part, but a number spent time in circular derivations. Some candidates gave a correct final answer through inconsistent or wrong derivations.*]

(b) For $r = 0$,



[7]

This part was well answered overall, although many students did not derive the simplest block diagram, frequently adding an extra reference signal and a sum element.

(c) (i) Phase margin: at crossover frequency we have about 135 degrees, thus a phase margin of 45 degrees. Gain margin: the phase is $-\pi$ at about 3 rad/s. At that frequency the magnitude is about -20 dB. Therefore, the gain margin is $\frac{1}{10^{-\frac{20}{20}}} = \frac{1}{10^{-1}} = 10$. [3]

(ii) Both phase and gain margins increase as k decreases and vice versa. [2]

(iii) $K(s) = \frac{k}{s}$ guarantees that $K(s)G(s)$ has a pole at 0, achieving perfect rejection $y(t) = 0$ of constant disturbances.

(The following detailed explanation was not requested in the paper).

Let $D(s)$ be the denominator of $G(s)$. For the second controller, the steady-state response is given by

$$\lim_{s\to 0^+} s\frac{1}{1+K(s)G(s)}\mathscr{L}(H(t)) = \lim_{s\to 0^+} s\frac{1}{1+\frac{k}{sD(s)}}\frac{1}{s} = \lim_{s\to 0^+} \frac{sD(s)}{sD(s)+k} = 0$$

where the last identity follows from the fact that the roots of $sD(s)+k$ are negative, by asymptotic stability. [4]

[*This part could be answered by considering the presence of an integrator in the controller but some students took the longer route of computing the complete response.*]

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) IFT is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\mathrm{e}^{i\omega t}d\omega$$

Therefore $x(t-T)$ is given by

$$x(t-T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \{X(\omega)\mathrm{e}^{-i\omega T}\}\mathrm{e}^{i\omega t}d\omega$$

So that

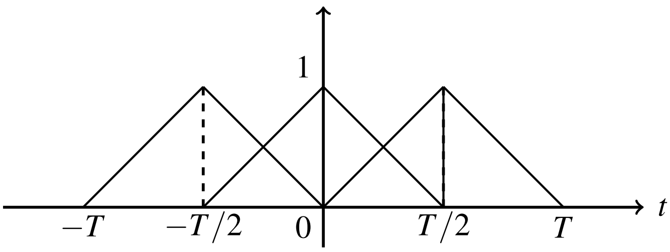
$$x(t-T) \longleftrightarrow X(\omega)\mathrm{e}^{-i\omega T}$$

Shifting in the time domain causes multiplication by a complex exponential in the frequency domain.

[4]

[*Well done by most candidates*]

(b) The signal $x(t)$ can be written as the sum of 3 triangular pulses, as shown here



From part (a) we therefore know that, if $\lambda(t)$ denotes the triangular pulse of height 1 and width T centred on the origin,

$$X(\omega) = \Lambda(\omega) + \Lambda(\omega)\mathrm{e}^{-i\omega T/2} + \Lambda(\omega)\mathrm{e}^{i\omega T/2} = \Lambda(\omega)[1+2\cos(\omega T/2)]$$

But we know that $\Lambda(\omega) = (T/2)\ \mathrm{sinc}^2\left(\frac{\omega T}{4}\right)$, so that

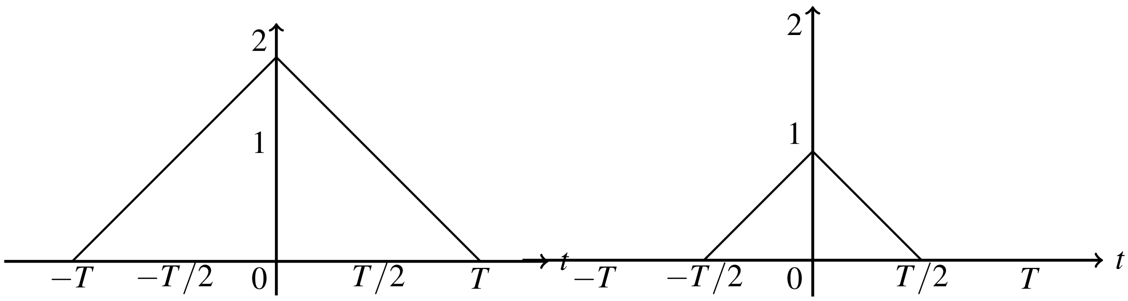
$$X(\omega) = (T/2)[1+2\cos(\omega T/2)]\ \mathrm{sinc}^2\left(\frac{\omega T}{4}\right)$$

Giving: $f(T,\omega) = (T/2)[1+2\cos(\omega T/2)]$.

[6]

[*This part produced some surprising ways of going wrong. Although most candidates split up the given function into the required 3 triangular pulses, some (not a negligible number) inverted the centre pulse, ie simply segmented the function by area.*]

(c) Note that the signal $x(t)$ can be expressed as the difference of two triangular pulses as shown below



Since $x(t)$ can therefore be written as $x(t) = 2\lambda(t/2) - \lambda(t)$, we can use the previous formula for the FT of $\lambda(t)$ to show that the FT of $x(t)$ can be written as

$$X(\omega) = 2T\ \mathrm{sinc}^2\left(\frac{\omega T}{2}\right) - \frac{T}{2}\ \mathrm{sinc}^2\left(\frac{\omega T}{4}\right)$$

[5]

[*Almost all candidates achieved full marks on this part. Though some subtracted a shifted triangle from the large triangle (ie again segmenting by area).*]

(d) It is easier to draw $X(\omega)$ from the form in part (b). Clearly the zeros occur when $1+2\cos(\omega T/2) = 0$ of $\mathrm{sinc}(\omega T/4) = 0$. In the first case we have $\cos(\omega T/2) = -1/2$, giving $\omega T/2 = 2\pi/3$ or $\omega = 4\pi/(3T)$. In the second case we have $\omega T/4 = \pi$, or $\omega = 4\pi/T$. The first zeros are therefore at $\omega = 4\pi/(3T)$. The graph is drawn below for $T = 1$: at $\omega = 0$ the height is $3T/2$, and our zeros occur at $\omega = (n\pi \pm \pi/3)/T$ and $\omega = n4\pi/T$. Therefore draw the graph by marking in the zeros, getting the mainlobe shape correct and indicating a decaying amplitude of the sidelobes.

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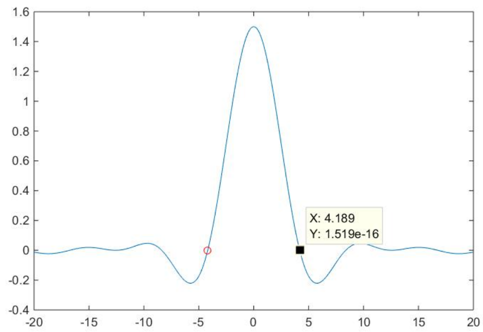


The width of the mainlobe is therefore $\frac{8\pi}{3T}$. [5]

[*This part presented more of a challenge. Many candidates simply found the zeros of the sinc function to find the width of the mainlobe – and many used part (c) instead of part (b) to draw the function, which made things harder.*]

(e) Parseval's Theorem enables us to relate the integrals in the time and frequency domains

$$\int_{-\infty}^{+\infty} |x(t)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

In this case, it is easier to do the integral in the time domain.

$$\int_{-\infty}^{+\infty} |x(t)|^2 dx = \int_{-T}^{-T/2} [2(1+t/T)]^2 dt \;+\; T \;+\; \int_{T/2}^{T} [2(1-t/T)]^2 dt$$

which can be written more concisely as

$$T + 2\int_0^{T/2} [1-2t/T]^2 dt = T - \left[\frac{T}{3}(1-2t/T)^3\right]_0^{T/2}$$

$$= T - \frac{T}{3}(0-1) = \frac{4T}{3}$$

Therefore giving $(1/2\pi)\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega = \frac{4T}{3}$. Since we have seen that the energy in the mainlobe dominates the energy in the whole spectrum, we can approximate the mainlobe energy with $\frac{4T}{3}$.

[5]

[*Again, this part proved a challenge. While many got this entirely correct, some tried to integrate in the frequency domain over the mainlobe width – which is hard!*]

5 (a) (i) To just avoid aliasing artefacts we need to sample at the Nyquist frequency, which is twice the largest frequency in the signal, ie $\omega_s = 2\omega_c$. Thus $(2\pi)/T = 2\omega_c$ so that $T = \pi/\omega_c$. [3]

[*Many people found this part easy, but a worrying number simply stated that the sampling frequency should be twice ω_c, rather than explicitly giving the sampling* period*, which was what was asked for.*]

(ii) The DFT is given by

$$X_k = \sum_{n=0}^{N-1} x_n e^{\frac{-2\pi jkn}{N}} \quad 0 \le k \le N-1$$

Multiply X_k by $e^{jkm2\pi/N}$ and sum over k:

$$\sum_{k=0}^{N-1} X_k e^{\frac{2\pi jkm}{N}} = \sum_{n=0}^{N-1} x_n \left\{ \sum_{k=0}^{N-1} e^{\frac{2\pi jk(m-n)}{N}} \right\}$$

To evaluate the quantity in curly brackets, note that this is a geometric progression, $S = \sum_{k=0}^{N-1} ar^k$ which is given by $S = a\frac{1-r^N}{1-r}$. In our case $a = 1$ and $r = e^{\frac{2\pi j(m-n)}{N}}$. If $m = n$ then clearly the sum is N. Since $e^{2\pi j(m-n)} = 1$, if $m \neq n$ the sum is

$$1\frac{1 - e^{2\pi j(m-n)}}{1 - e^{\frac{2\pi j(m-n)}{N}}} = 0$$

thus, we have

$$\sum_{k=0}^{N-1} X_k e^{\frac{2\pi jkm}{N}} = Nx_m$$

which gives us the formula for the inverse DFT. [9]

[*This part presented trouble for many. Some had clearly revised the derivation and rattled it off easily, some however tried to derive the IDFT via a continuous inverse FT.*]

(b) (i) The block diagram of the receiver consisting of a product modulator followed by a low pass-filter is shown below. As a perfect copy of the carrier is available, $\phi = 0$. The output of the product modulator is

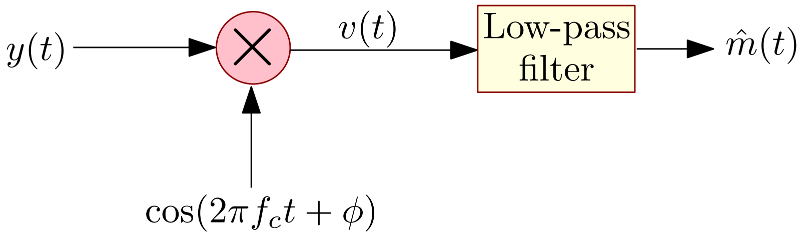
$$v(t) = y(t) \cdot \cos(2\pi f_c t) = \alpha m(t)\cos^2(2\pi f_c t) = \alpha m(t)\left(\frac{1+\cos(4\pi f_c t)}{2}\right)$$

To recover $m(t)$, we can use a low pass filter whose gain is a constant equal to $\frac{2}{\alpha}$

 (cont.

in the frequency band $[-W, W]$, and zero outside. A non-ideal low pass filter would have a transition band, i.e, a gentler roll-off to zero. [8]

[*Generally well done*]



(ii) Here the phase $\phi = -45^\circ = -\frac{\pi}{4}$. Therefore the product modulator output is now (using $2(\cos A + \cos B) = [\cos(A+B)/2][\cos(A-B)/2]$)

$$v(t) = \alpha m(t)\cos(2\pi f_c t)\cos(2\pi f_c t - \frac{\pi}{4}) = \frac{1}{2}\alpha m(t)\left(\cos(\frac{\pi}{4}) + \cos(4\pi f_c t - \frac{\pi}{4})\right)$$

The output of the low-pass filter (with the same gain as in part (b)(i)) is therefore

$$\hat{m}(t) = m(t)\cos(\pi/4) = \frac{m(t)}{\sqrt{2}}$$

Note that the receiver cannot compensate for the $1/\sqrt{2}$ factor as it does not know the phase la [5]

[*Many lost marks here by not reading the question properly, ie* if the carrier is used in the receiver of part (b)(i), *so failed to apply the* $\frac{2}{\alpha}$ *gain. One or two reverted to Laplace transforms and got into quite a mess.*]

6 (a) (i) The optimal detection rule is to choose the constellation symbol closest to the observed Y. Thus

$$\hat{X} = \begin{cases} -2A & \text{if } -\infty < Y \le -A \\ 0 & \text{if } -A < Y \le A \\ 2A & \text{if } A < Y < \infty \end{cases}$$

[3]

[*Mostly done well. Though a few candidates merely drew an unclear diagram with no explanation.*]

(ii) The probability of error when $-2A$ is sent is

$$P(\hat{X} \ne -2A \mid X = -2A) = P(Y > -A \mid X = -2A) = P(X + N > -A \mid X = -2A) =$$

$$P(-2A+N > -A \mid X=-2A) = P(N > A) = P\left(\frac{N}{\sigma} > \frac{A}{\sigma}\right) = Q\left(\frac{A}{\sigma}\right)$$

By symmetry, the probability of error when $2A$ is transmitted is the same.
The probability of error when 0 is sent is

$$P(\hat{X} \neq 0 \mid X=0) = P(|Y| > A \mid X=0) = P(|X+N| > A \mid X=0)$$
$$= P(|N| > A) = 2Q\left(\frac{A}{\sigma}\right)$$

As all the symbols are equally likely, the overall probability of detection error is

$$P_e = \frac{1}{3}P(\hat{X} \neq -2A \mid X=-2A) + \frac{1}{3}P(\hat{X} \neq 0 \mid X=0) + \frac{1}{3}P(\hat{X} \neq 2A \mid X=2A)$$
$$= \frac{1}{3}\left(Q\left(\frac{A}{\sigma}\right) + 2Q\left(\frac{A}{\sigma}\right) + Q\left(\frac{A}{\sigma}\right)\right) = \frac{4}{3}Q\left(\frac{A}{\sigma}\right).$$

[6]

[*This was a very difficult question to mark. There were many very good answers, but also a fair number of answers which had incorrect conditional probablility expressions – of course, all ended up with the given solution.*]

(iii) The average energy per symbol is

$$E_s = \frac{((-2A)^2 + 0^2 + (2A)^2)}{3} = \frac{8A^2}{3} \quad \Rightarrow A^2 = \frac{3E_s}{8}.$$

Hence

$$P_e = \frac{4}{3}Q\left(\frac{A}{\sigma}\right) = \frac{4}{3}Q\left(\sqrt{\frac{3E_s}{8\sigma^2}}\right).$$

[3]

[*Mostly well done.*]

(iv) Using the given approximation, the P_e above can be written as

$$P_e = \frac{4}{3}Q\left(\sqrt{\frac{3E_s}{8\sigma^2}}\right) \approx \frac{2}{3}\mathrm{e}^{-\frac{3E_s}{16\sigma^2}}.$$

Setting this, equal to 10^{-5} and solving we get

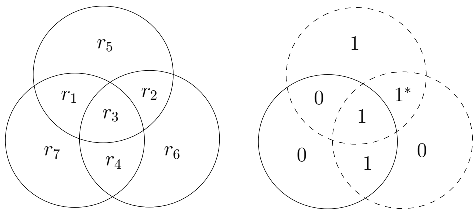
$$\frac{E_s}{\sigma^2} = \frac{16}{3}\ln\left(\frac{2\times 10^5}{3}\right) = 59.24 \quad (\text{or } 10\log_{10}59.24) = 17.73 \text{ dB})$$

[3]

[*Mostly well done.*]

 (cont.

(b) (i) The Hamming code can be decoded using the "parity circles" in Fig. (a) below. If *no errors are made*, each of the three circles should have parity 0 – this is because $r_1, \ldots, r_4$ are the four source bits and $r_5 = r_1 \oplus r_2 \oplus r_3$, $r_6 = r_2 \oplus r_3 \oplus r_4$, and $r_7 = r_1 \oplus r_3 \oplus r_4$.



(a) (b)

To decode $\mathbf{r} = [0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 0]$, fill in the bits in circles as shown in Fig. (b). The dashed circles indicate those for which the parity is 1. The decoder flips one bit so that the parity of the dashed circles becomes 0, *without* changing the 0 parity of the solid circle. This bit is r_2, which should be flipped from 1 to 0. Therefore the decoded codeword is $[0,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0]$. [5]

(ii) The probability of correct decision for a BSC with crossover probability ε is the probability that the channel flips one or zero bits. Hence

$$P_{correct} = \binom{7}{0}(1-\varepsilon)^7 + \binom{7}{1}\varepsilon(1-\varepsilon)^6$$

The probability of decoding error is

$$P_e = 1 - P_{correct} = 1 - \binom{7}{0}(1-\varepsilon)^7 - \binom{7}{1}\varepsilon(1-\varepsilon)^6$$

For $\varepsilon = 0.1$, $P_e = 0.1497$. [5]

[*If candidates did not run out of time, most achieved full marks on part (b).*]

END OF PAPER

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