Solutions to IB Mathematical Methods, 2014

Section A

1. *Divergence and Gauss*

(a) We integrate $\underline{I}$ over the hemispherical surface S, by writing the differential area element $d\underline{S} = \underline{e}_r dS$, and

$$dS = r\sin\theta \; d\theta \; r \; d\phi$$

Most students did not remember that it is necessary to multiply r by $\sin\theta$. In addition, even though this is an area, most forgot to check that it had the correct dimensions. Always check left and right hand side dimensions - a quick an easy way to find errors.

The integration is straightforward:

$$\begin{aligned} \underline{I} \cdot d\underline{S} &= \int_0^{2\pi} \int_0^{\pi/2} \frac{I_0}{r^2} \cos\theta \; r^2 \sin\theta \; d\theta \; d\phi \\ &= 2\pi I_0 \int_0^{\pi/2} \frac{\sin 2\theta}{2} d\theta = 2\pi I_0 \left[-\frac{\cos 2\theta}{4} \right]_0^{\pi/2} \\ &= \frac{\pi}{2} I_0 (1 - \cos 2\frac{\pi}{2}) = \pi I_0 \end{aligned}$$

[8]

(b) We now integrate $\underline{I}$ over the solid angle θ_0 determined by R and H:

$$\begin{aligned} \underline{I} \cdot d\underline{S} &= \int_0^{\theta_0} \int_0^{2\pi} \frac{I_0}{r^2} \cos\theta \; r^2 \sin\theta \; d\theta \; d\phi \\ &= \frac{\pi}{2} I_0 (1 - \cos 2\theta_0) = \frac{\pi}{2} I_0 (1 - 2\frac{R^2}{R^2 + H^2}) \\ &= \pi I_0 \frac{R^2}{R^2 + H^2} \end{aligned}$$

where we used the fact that $\dfrac{1 - \cos 2\theta_0}{2} = \sin^2\theta = \dfrac{R^2}{R^2 + H^2}$.

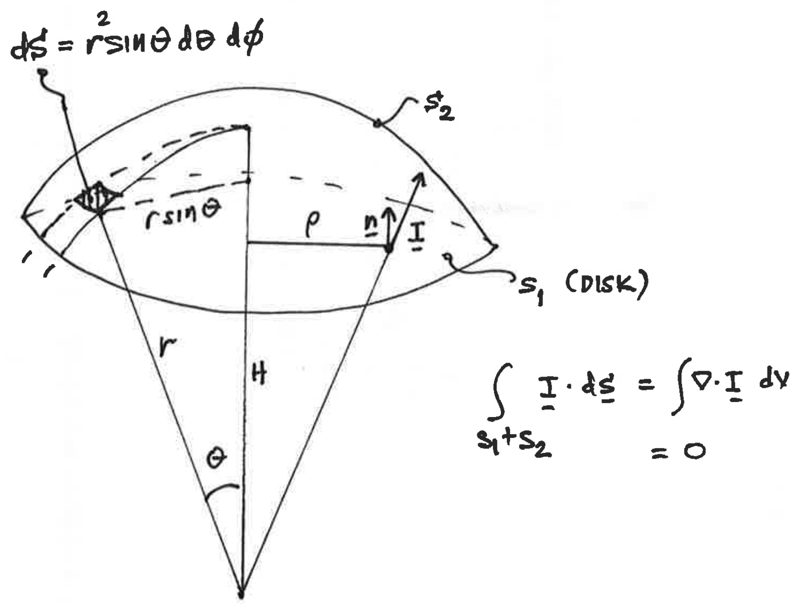
A number of students correctly used the ratio of solid angles between the hemisphere and the solid angle to calculate the total flux normal to the region, which also works. [8]

(c) We calculate the integral to the surface of the disk directly, where $d\underline{S} = \rho \; d\phi \; d\rho \; \underline{n}$, $\rho = H\tan\theta$ is a radial coordinate for the disk, ϕ the angle around the disk, and $\underline{n}$ is the normal to the disk, so that $\underline{e}_r \cdot \underline{n} = \cos\theta$

$$
\begin{aligned}
\underline{I}\cdot d\underline{S} &= \int_0^R \int_0^{2\pi} \frac{I_0}{\rho^2+H^2}\cos\theta\ \underline{e}_r\cdot\underline{n}\ \ \rho\ d\phi\ d\rho \\
&= 2\pi I_0 \int_0^R \frac{\cos^2\theta}{\rho^2+H^2}\ \rho d\rho \\
&= 2\pi I_0 \int_0^R \frac{\cos^2\theta}{\rho^2+H^2}\ H\tan\theta \frac{H}{\cos^2\theta} d\theta \\
&= 2\pi I_0 \int_0^{\theta_0} \cos\theta\sin\theta\ d\theta = \\
&= 2\pi I_0 \left[-\frac{\cos 2\theta}{4}\right]_0^{\theta_0} = \frac{\pi}{2} I_0 (1-\cos 2\theta_0) \\
&= \pi I_0 \frac{R^2}{R^2+H^2}
\end{aligned}
$$

This means the result is the same whether we calculate the integral over the disk on the projected solid angle or the shell encompassed by the solid angle. This is true because the power $\underline{I}\cdot d\underline{S}$ is determined by the dotted product of the surface area and the flux: any surface with the same projected area will produce the same value of power.

Another way to look at it is that the divergence of the flux is zero, so that whatever comes in through one surface over the same solid angle must come out radially somewhere else, and there is no accumulation. [9]



dS = r² sinθ dθ dφ
S2
r sinθ
ρ
n
I
S1 (DISK)
r
H
θ
∫ I · dS = ∫ ∇ · I dV
S1+S2
= 0

2. *Curl, div and grad*

(a) Using the expression for curl in cylindrical coordinates, we have:

$$\nabla \times \underline{V} = \begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & v_\theta & v_z \end{vmatrix} = \frac{\partial}{\partial r}\left(r\frac{\Gamma}{2\pi r}\right) - \frac{\partial}{\partial \theta}\left(-\frac{Q}{2\pi r}\right) = 0$$

The most common mistake in this step was to forget the differentiation in cylindrical coordinates in r includes multiplication by r prior to differentiation. [4]

(b)(i) Here we take the *actual* paths given in the problem and integrate directly, according to the definition. The circular paths only have the tangential component V_θ, $d\underline{\ell} = \pm r\, d\theta \underline{e}_\theta$, and the radial paths only have V_r and $d\underline{\ell} = \pm dr\ \underline{e}_r$. For the path in (i), we have

$$\oint \underline{V} \cdot d\underline{\ell} = \int_0^{2\pi} \frac{\Gamma}{2\pi\cancel{(2a)}}\cancel{(2a)}\, d\theta + \int_0^{2\pi} \frac{\Gamma}{2\pi\cancel{(a)}}\cancel{(a)}\, d\theta + \int_{2a}^{a} \frac{-Q}{2\pi r}\, dr + \int_a^{2a} \frac{-Q}{2\pi} r\, dr = 0$$

This is of course expected, since the curl is zero *away from the singularity* at the origin. [3]

(b)(ii) For the circle around the singularity, we have:

$$\oint \underline{V} \cdot d\underline{\ell} = \int_0^{2\pi} \frac{\Gamma}{2\pi\cancel{(2a)}}\cancel{(2a)}\, d\theta = \Gamma$$

[3]

(c) So now we have two paths, producing different path integrals. For closed path integrals not including singularity at the origin, the integral always gives zero. However, for any path including the origin, we will have:

$$\oint \underline{V} \cdot d\underline{\ell} = \nabla \times \underline{V} \cdot d\underline{S} = \Gamma$$

as illustrated by the integrals in (a) and (b). In order to make the two definitions compatible, we write:

$$\nabla \times \underline{V} \ = \Gamma\delta(\underline{r})$$

where $\delta(r)$, the delta function, which is zero everywhere but at the origin, and whose integral over the surface is unity.

The statement expresses the fact that the flow is an ideal vortex (and sink), where the flow is well behaved everywhere, but whose source of energy and movement is concentrated at a single spot.

This is a means of simplifying many phenomena in fields such as flow and electromagnetism, where sources or sinks of vorticity or flow are identified such as to generate a particular fields, and eliminating the complications associated with the actual sources, such as walls, charges and viscous effects. [2]

(d) The divergence of the field is given as:

$$\begin{aligned} \nabla \cdot \underline{V} &= \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{-Q}{2\pi r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\Gamma}{2\pi r}\right) = 0 \end{aligned}$$

We have a singularity at the origin. From Gauss' theorem, we expect, for a circle around the origin:

$$\begin{aligned} \oint \nabla \cdot \underline{V}\, dV &= \int \underline{V} \cdot \underline{n}\, d\underline{S} \\ &= \int_0^{2\pi} V_r\, d\theta = \int_0^{2\pi} \frac{-Q}{2\pi r}\, r d\theta = -Q \end{aligned}$$

which is the strength of the sink. Therefore, we should define:

$$\nabla \cdot \underline{V} = -Q\delta(\underline{r})$$

so as to be consistent everywhere in the flow.

Most candidates did not get either (c) or (d) correctly on this front. [4]

(e) We can obtain a potential flow field ϕ from the integration of velocity

$$\begin{aligned}
V_r &= \frac{\partial \phi}{\partial r} \to \phi = \int \frac{-Q}{2\pi r}\, dr = -\frac{-Q}{2\pi} \ln r + f(\theta) \\
V_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} = \frac{1}{r} f'(\theta) \\
\phi &= -\frac{-Q}{2\pi} \ln r + \frac{\Gamma}{2\pi}\theta \\
2\pi\phi &= -Q \ln r + \Gamma\theta \\
\frac{2\pi\phi}{Q} &= -\ln r + \frac{\Gamma}{Q}\theta
\end{aligned}$$

[5]

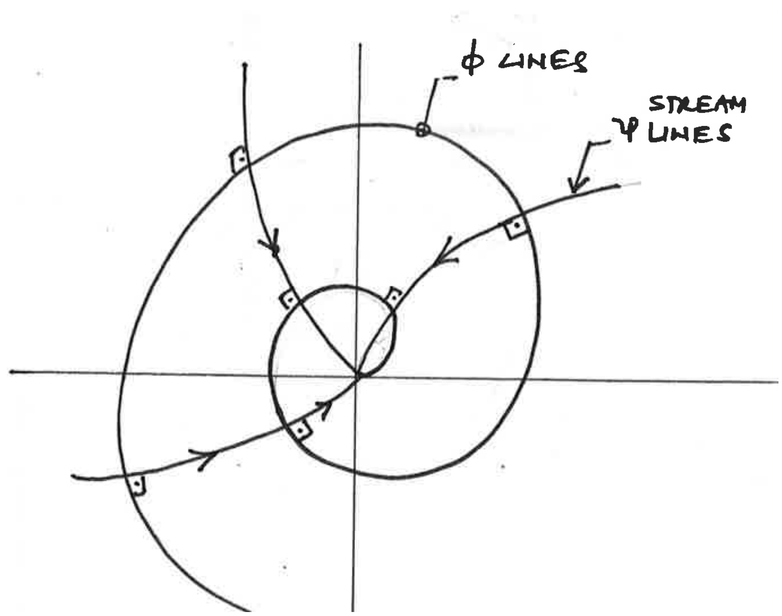
(f) The flow is a combination of an ideal sink (with streamlines flowing towards the origin) and an ideal vortex (with streamlines flowing around a circle). The ratio Γ/Q determines the inclination of the streamlines relative to an ideal sink.

The streamlines can either be obtained as perpendicular to the potential flow, or directly from:

$$\begin{aligned}
\frac{dr}{V_r} &= \frac{r d\theta}{V_\theta} \\
-\frac{dr}{Q} &= \frac{r d\theta}{\Gamma} \\
d \ln r &= -\frac{Q}{\Gamma} d\theta \\
2\pi\psi &= \ln r + \frac{Q}{\Gamma}\theta
\end{aligned}$$

For constant values of ψ, we have $r = e^{(2\pi\psi)} e^{-(Q/\Gamma)\theta}$, so that the radius decreases with θ and with Q/Γ, with a limit of zero for increasing θ. This is therefore a spiral which curls to zero, faster depending on the ratio of flow to circulation.

Many students plotted the spiral represented by the potentials in (f). These were considered as correct as well, even if not labelled properly. [4]


ϕ LINES
STREAM
ψ LINES

3. *PDE separation of variables*

(a) We assume the solution is separable in the form of a product $T(x, y) = X(x)Y(y)$, and differentiate and substitute into the original equation to get:

$$\begin{aligned}
\lambda_x X''Y + \lambda_y Y''X &= 0 \\
\frac{X''}{X} + \beta^2 \frac{Y''}{Y} &= 0 \\
\frac{X''}{X} = -\beta^2 \frac{Y''}{Y} &= m^2
\end{aligned}$$

We solve each equation separately, with the appropriate boundary conditions:

$$\begin{aligned}
X'' - m^2 X &= 0 \\
X &= Ae^{mx} + Be^{-mx} \\
Y'' + (m/\beta)^2 Y &= 0 \\
Y &= Ce^{im/\beta y} + De^{-im/\beta y}
\end{aligned}$$

(we can equally recognise the boundary conditions quicker and choose a basis of sinh and cosh to speed things up. From the boundary conditions:

$$T(x, 0) = (Ae^{mx} + Be^{-mx})((C + D)\cos((m/\beta)0) + (C - D)\sin((m/\beta)0)) = 0$$

we have $C = D$.

$$T(x, L) = (Ae^{mx} + Be^{-mx})(2C)\sin(m/\beta L) = 0$$

This is only possible for $mL/\beta = \pm n\pi$.

For the boundary conditions in x, we have:

$$T(-L/2) = (Ae^{-mL/2} + Be^{mL/2})(2C)\sin(n\pi/Ly) = \sin(\pi/Ly)$$

$$T(L/2) = (Ae^{mL/2} + Be^{-mL/2})(2C)\sin(n\pi/Ly) = \sin(\pi/Ly)$$

$n = 1$, $m = \pi\beta/L$.

$$\begin{aligned}
2C(Ae^{-\pi\beta/2} + Be^{\pi\beta/2}) &= 1 \\
2C(Ae^{\pi\beta/2} + Be^{-\pi\beta/2}) &= 1
\end{aligned}$$

From which we have

$$2(AC) = 2(BC) = \sinh(\pi\beta/2)/\sinh(\pi\beta)$$

so that

$$\begin{aligned}
T(x, y) &= 2\frac{\sinh(\pi\beta/2)}{\sinh(\pi\beta)}\cosh(\pi\beta x/L)\sin(\pi y/L) = \\
&= \frac{1}{\cosh(\pi\beta/2)}\cosh(\pi\beta x/L)\sin(\pi y/L)
\end{aligned}$$

[11]

(b) This is straightforward differentiation:

$$\begin{aligned}
q_x &= -\lambda_x \frac{\partial T}{\partial x} = -\lambda_x \frac{\pi\beta}{L}\sinh(\pi\beta x/L)\sin(\pi y/L) \\
q_y &= -\lambda_y \frac{\partial T}{\partial y} = -\lambda_y \frac{\pi}{L}\cosh(\pi\beta x/L)\cos(\pi y/L)
\end{aligned}$$

[3]

(c) The divergence is given by:

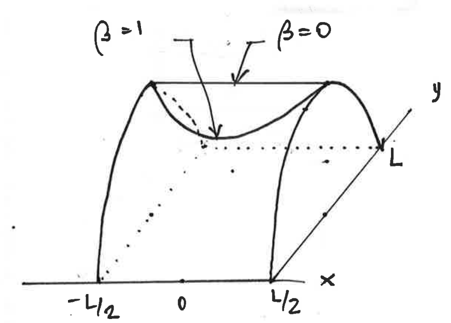
$$\nabla \cdot \underline{q} = -\lambda_x \frac{\partial^2 T}{\partial x^2} - \lambda_y \frac{\partial^2 T}{\partial y^2} =$$

$$= -\lambda_x \left(\frac{\pi\beta}{L}\right)^2 \cosh(\pi\beta x/L)\sin(\pi y/L) + \lambda_y \left(\frac{\pi}{L}\right)^2 \cosh(\pi\beta x/L)\sin(\pi y/L) = 0$$

[3]

(d) The sketch gave all the most trouble. It is easy to recognise that the variation in the y-direction is sinusoidal, but the domain is only a quarter wave. In the x-direction, the sinusoidal is modulated by the amplitude given by the magnitude of β, symmetrically about the axis.

For $\beta = 0$, there is no variation in x, and no conduction of heat in the x-direction. For $\beta = 1$, the material is homogeneous with constant conductivity in all directions. The solution looks like a 3D saddle.



[8]

Section B

4. *Probability*

(a) The probability that the measurement is above a threshold is given by

$$\mathcal{F}(x) = \int_{\tau}^{\infty} \mathcal{N}(z; x, \sigma^2) dz$$

Re-expressing this in terms of the standard Gaussian yields

$$\mathcal{F}(x) = \int_{(\tau - x)/\sigma}^{\infty} \mathcal{N}(z; 0, 1) dz$$

Since the Gaussian is symmetric this can be written as

$$\mathcal{F}(x) = \int_{-\infty}^{(x-\tau)/\sigma} \mathcal{N}(z; 0, 1) dz$$

[5]

(b) Simply putting the numbers into the expression yields

$$\mathcal{F}(0.7) = \int_{-\infty}^{-0.316228} \mathcal{N}(z, 0, 1) dz = 0.37591$$

[2]

(c)(i) The tests are all independent. Averaging the five tests will reduce the variance by a factor of five. Thus the resulting variance is 0.02. Substituting this expression in yields

$$\mathcal{F}(0.7) = \int_{-\infty}^{-0.707107} \mathcal{N}(z, 0, 1) dz = 0.23975$$

[6]

(c)(ii) The probability of the final classification result being incorrect is the probability that 3 or more results from the individual tests are incorrect. This can be written as

$$(0.37591)^5 + 5 \times (0.37591)^4 \times 0.62409 + 10 \times (0.37591)^3 \times (0.62409)^2 =$$
$$0.0075062 + 0.062309 + 0.20689 = 0.27671$$

[7]

(c)(iii) Averaging the raw scores from the tests yielded a lower probability of error than using the outcome of the tests. This is as expected because information is lost (how far the result was from the threshold) if the classification outcomes are used. When only a single test is performed the outcomes are the same. For all other cases averaging the raw results will be better. In addition when an even number of tests is performed there can be ties - equal number of correct and incorrect results. [5]

5. *Subspaces and Inversion*

(a) For a matrix to be invertible its determinant must be non-zero, i.e. it must be of full rank. [2]

(b) For $a = 0$ the following expression is being solved

$$\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 0 \\ 0 & 3 & 2 \end{bmatrix} \mathbf{x}$$

The design matrix is of full-rank, hence the matrix is invertible. It is simple to solve this expression in two stages. Considering the first two elements only

$$\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This yields $x_1 = 2, x_2 = 1$. Substituting this yields the final solution

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

with a design cost of $k\sqrt{5}$. [5]

(c)(i) For the design matrix to be non-invertible requires that the matrix is not of full rank. Re-arranging (subtracting 0.5 row 1 from row 2)

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & a \\ 0 & 3 & 2 \end{bmatrix}$$

Clearly set $a = 2$ shows that the matrix is not of full rank. [4]

(c)(ii) The solution to part (b) is clearly also a possible solution to the case when $a = 2$. The null-space for this matrix is perpendicular to the row-space. Taking the cross product

$$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix}$$

Thus the general solution to this design problem is

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

[8]

(c)(iii) To minimise the design cost requires minimising the magnitude of $\mathbf{x}$. Consider minimising the squared magnitude of $\mathbf{x}$

$$(2 + 2\lambda)^2 + (1 - 2\lambda)^2 + 9\lambda^2 = 17\lambda^2 + 4\lambda + 5$$

Differentiating and equating to zero yields

$$34\lambda + 4 = 0$$

Thus $\lambda = -2/17$. The value of $\mathbf{x}$ in then

$$\mathbf{x} = \begin{bmatrix} 2 - 4/17 \\ 1 + 4/17 \\ -6/17 \end{bmatrix}$$

The cost of this design is $2.1828k$. [6]

6. *Eigenvectors and EigenValues*

(a) The expression that must be satisfied by an eigenvector is

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

[2]

(b) Producting out the expression

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{12}\mathbf{x}_2 \\ \mathbf{A}_{22}\mathbf{x}_2 \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

(i) If $\mathbf{x}_2 \neq \mathbf{0}$ then an eigenvalue of $\mathbf{A}$ must satisfy

$$\mathbf{A}_{22}\mathbf{x}_2 = \lambda\mathbf{x}_2$$

Thus it must be an eigenvalue of $\mathbf{A}_{22}$. [4]

(ii) If $\mathbf{x}_2 = \mathbf{0}$ then the following expression is satisfied

$$\mathbf{A}_{11}\mathbf{x}_1 = \lambda\mathbf{x}_1$$

Thus it must be an eigenvalue of $\mathbf{A}_{11}$. [4]

Since these two cases cover all possibilities then eigenvalues of $\mathbf{A}$ must be eigenvalues of either $\mathbf{A}_{11}$ or $\mathbf{A}_{22}$. Any eigenvector unique to $\mathbf{A}_{11}$ must have $\mathbf{x}_2 = \mathbf{0}$. [3]

(c)(i) From the previous section the eigenvalues of $\mathbf{A}$ must be the eigenvalues of either

$$\mathbf{A}_{11} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

which has eigenvalues 6 and 4, or

$$\mathbf{A}_{22} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

which has eigenvalues 3 and 1. [4]

(c)(ii) The magnitude and direction will be dominated by the largest eigenvalue. The expression for raising a square matrix to the power n is

$$\mathbf{A}^n = \mathbf{X}\mathbf{\Lambda}^n\mathbf{X}^{-1}$$

where $\mathbf{X}$ is the matrix of the eigenvectors and $\mathbf{\Lambda}$ the matrix where the leading diagonals are the eigenvalues. Note in contrast to a symmetric matrix the eigenvectors are not orthogonal to one another, so the transpose is not the inverse.

The largest eigenvalue is 6 and unique to $\mathbf{A}_{11}$ so $\mathbf{x}_2 = \mathbf{0}$. The eigenvector associated with this must satisfy

$$\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \mathbf{x}_1 = 6\mathbf{x}_1$$

The eigenvector is thus

$$\mathbf{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$\mathbf{y} \propto 6^n \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The value of the the scaling constant is given by the first element (assuming rank-ordered) of

$$\mathbf{X}^{-1}\begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}$$

Computing this inverse was not expected.

Various other forms were acceptable, provided some discussion of the inverse, scaling with the largest eigenvalue, and the need to find the value. [8]