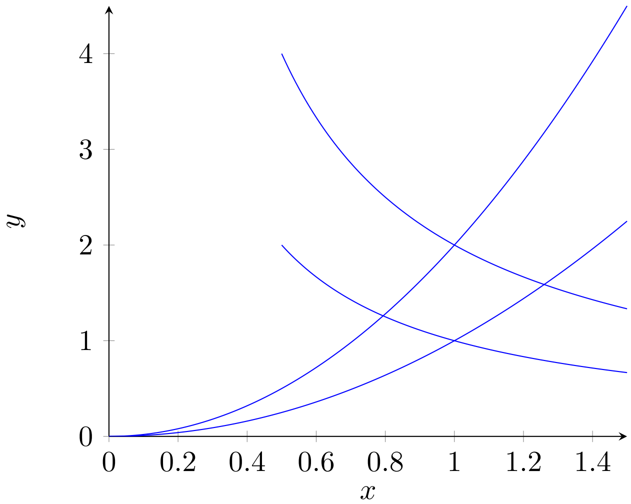
Section A.

1. *Variable change, divergence and curl*

(a)

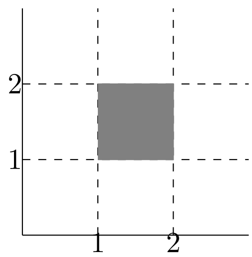


[3]

(b)

$$\left.\begin{array}{l} u = x^2 y \\ v = y/x^2 \end{array}\right\} \rightarrow \quad \left.\begin{array}{l} uv = y^2 \\ u/v = x^4 \end{array}\right\} \rightarrow \quad \begin{array}{l} y = \sqrt{uv} \\ x = (u/v)^{1/4} \end{array}$$

$$\begin{array}{ll} y = x^2 \rightarrow v = 1 & y = 1/x^2 \rightarrow u = 1 \\ y = 2x^2 \rightarrow v = 2 & y = 2/x^2 \rightarrow u = 2 \end{array}$$



We need to find the Jacobian:

$$\begin{aligned} J &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= \frac{1}{4}\frac{1}{u^{3/4}v^{1/4}}\frac{1}{2}\frac{u^{1/2}}{v^{1/2}} + \frac{1}{4}\frac{u^{1/4}}{v^{5/4}}\frac{1}{2}\frac{v^{1/2}}{u^{1/2}} = \\ &= \frac{1}{8}\frac{1}{u^{1/4}v^{3/4}} + \frac{1}{8}\frac{1}{u^{1/4}v^{3/4}} = \\ &= \frac{1}{4}\frac{1}{u^{1/4}v^{3/4}} \end{aligned}$$

$$\begin{aligned} I = \int\int_R \frac{y}{x}\, dx\, dy = \int\int_R \frac{\sqrt{uv}}{(u/v)^{1/4}}|J|\, du\, dv = \\ = \int\int_R \frac{u^{1/4}v^{3/4}}{1}\frac{1}{4}\frac{1}{u^{1/4}v^{3/4}} = \frac{1}{4} \end{aligned}$$

Some students calculated the inverse Jacobian, and directly substituted the values of u and v as a function of x and y to obtain the same result (because it cancels out). Although acceptable, the solution above is the correct one. [15]

(c) *Stokes*

$$\oint \mathbf{B} \cdot d\mathbf{r} = \int \nabla \times \mathbf{B} \cdot d\mathbf{A}$$

But $\mathbf{B}$ is in $x - y$ plane, so that $\nabla \times \mathbf{B}$ is in z direction:

$$\begin{aligned}
\int \nabla \times \mathbf{B} \cdot d\mathbf{A} &= \int_R [\nabla \times \mathbf{B}]_z \, dxdy \\
\mathbf{B}_z &= \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0 - \frac{2y}{x} \\
\Gamma = -2 \int \int_R \frac{y}{x} \, dx \, dy &= -2I = -\frac{1}{2}
\end{aligned}$$

A majority of students did use Stokes to obtain the integral above, and related it to the previously calculated value of I. Minor issues appeared with sorting out the correct sign. [7]

2. *Gradients and divergence*

(a) *Divergence*

$$\begin{aligned}\mathbf{u} = \nabla g &= 3z^2(x^2+y^2)\mathbf{k} + 2xz^3\mathbf{i} + 2yz^3\mathbf{j} \\ &= 2z^3(x\mathbf{i}+y\mathbf{j}) + 3z^2(x^2+y^2)\mathbf{k}\end{aligned}$$

In polar coordinates,

$$\mathbf{u} = 2z^3\rho\ \hat{\mathbf{e}}_\rho + 3z^2\rho^2\ \hat{\mathbf{e}}_k$$

$$\begin{aligned}\nabla\cdot\mathbf{u} &= \frac{\partial}{\partial x}(2xz^3) + \frac{\partial}{\partial y}(2yz^3) + \frac{\partial}{\partial z}(3z^2(x^2+y^2)) = \\ &= 2z^3 + 2z^3 + 6z(x^2+y^2) = \\ &= 4z^3 + 6z(x^2+y^2) && \text{Cartesian} \\ &= 4z^3 + 6z\rho^2 && \text{Polar}\end{aligned}$$

[5]

(b) Gauss:

$$\oint \mathbf{u}\ \cdot\ d\mathbf{A} = \int_V \nabla\cdot\mathbf{u}\, dV$$

In polar coordinates,

$$\begin{aligned}\int_V \nabla\cdot\mathbf{u}\ dV &= \int_{-1}^{1}\int_0^1 [4z^3 + 6z\rho^2]\ 2\pi\rho\ d\rho\ dz \\ &= \int_{-1}^{1} 4z^3\ dz \int_0^1 2\pi\rho\ d\rho + \int_{-1}^{1} 6z\ dz \int_0^1 2\pi\rho^3\ d\rho\end{aligned}$$

Both integrals in z are odd in z, and so integrate to zero:

$$\oint \mathbf{u}\ \cdot\ d\mathbf{A} = 0$$

Attempts to integrate the fluxes using the full area can succeed, if care is taken to consider the full area, but that approach produced errors, as some students did not consider the cancelation of top and bottom of the cylinder.

(c) [6]

$\oint \mathbf{u}\ \cdot\ d\mathbf{A} = 0$ because $\nabla\times\mathbf{u} = \nabla\times(\nabla g) = 0$ (cf. vector identity $\nabla\times(\nabla\phi) = 0$ for any scalar function ϕ. Alternatively, one can evaluate $\nabla\times\mathbf{u}$ using the definition of curl, to find $\nabla\times\mathbf{u} = 0$.

[4]

(d) *Stokes*

From Stokes' theorem,

$$\Gamma = \oint_C \mathbf{u}\cdot d\mathbf{r} = \int_{S^*} \nabla\times\mathbf{u}\cdot d\mathbf{A}$$

where S^* is the surface that spans C. But $\nabla\times\mathbf{u} = 0$, so $\Gamma = 0$. [4]

(e) *Gradient potential*

$$\begin{aligned}\mathbf{v} = g\nabla g &= g[\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}] = \\ &= \frac{\partial}{\partial x}(\frac{1}{2}g^2)\mathbf{i} + \frac{\partial}{\partial y}(\frac{1}{2}g^2)\mathbf{j} + \frac{\partial}{\partial z}(\frac{1}{2}g^2)\mathbf{k} = \\ &= \nabla(\frac{1}{2}g^2)\end{aligned}$$

The scalar potential of $\mathbf{v}$ is defined by $\mathbf{v} = -\nabla\phi$ (or it could be $-\phi$), so that

$$\phi = \pm\frac{1}{2}g^2$$

We cannot introduce a vector potential for $\mathbf{u}$ because the field is not solenoidal, i.e. $\nabla \cdot \mathbf{u} \neq 0$. [6]

3. *Wave equation*

(a). We have for the homogeneous solution p, using the intermediate variable $\zeta = x \pm ct$:

$$\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{df}{d\zeta}\frac{\partial \zeta}{\partial x} + \frac{dg}{d\zeta}\frac{\partial \zeta}{\partial x} \\
\frac{\partial p}{\partial x} &= f' + g' \\
\frac{\partial p}{\partial t} &= \frac{df}{d\zeta}\frac{\partial \zeta}{\partial t} + \frac{dg}{d\zeta}\frac{\partial \zeta}{\partial t} \\
\frac{\partial p}{\partial t} &= cf' - cg' \\
f'' + g'' &= c^2(f'' + g'')
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 p}{\partial x^2} &= \frac{d^2 f}{d\zeta^2}\frac{\partial \zeta}{\partial x} + \frac{d^2 g}{d\zeta^2}\frac{\partial \zeta}{\partial x} \\
\frac{\partial^2 p}{\partial x^2} &= f'' + \ g'' \\
\frac{\partial^2 p}{\partial t^2} &= \frac{d^2 f}{d\zeta^2}\frac{\partial \zeta}{\partial x} + \frac{d^2 g}{d\zeta^2}\frac{\partial \zeta}{\partial t} \\
\frac{\partial^2 p}{\partial t^2} &= c^2 f'' + c^2 g''
\end{aligned}$$

QED.

[4]

(b). Using $p = X(x)T(t)$:

$$\begin{aligned}
\frac{\partial^2 p}{\partial x^2} &= X''T \\
\frac{\partial^2 p}{\partial t^2} &= T''X \\
c^2 X''T &= T''X \\
c^2 \frac{X''}{X} = \frac{T''}{T} &= -\omega^2
\end{aligned}$$

We then solve two differential equations to yield:

$$X = A\cos kx + B\sin kx \qquad T = C\cos\omega t + D\sin\omega t$$

where $k = \omega/c$.

[5]

(c) Since we are seeking a solution that vanishes at $x = a$, but not at $x = 0$, we choose to retain the cosine term (other combinations will give different intermediate solutions, with the same final solution), and incorporate the constant A into the other constants:

$$\begin{aligned}
p(a,t) = \cos(ka)(C\cos\omega t + D\sin\omega t) &= 0 \\
k_n a &= (2n+1)\pi/2 \\
\omega_n = k_n c &= (c/a)(2n+1)(\pi/2) \\
p(x,t) = \cos(\omega_n x/c)(C\cos\omega_n t &+ D\sin\omega_n t)
\end{aligned}$$

Check:

$$\begin{aligned}
p(0,t) &= \cos(\omega_n 0/a)(C\cos\omega_n t + D\sin\omega_n t) \\
p(a,t) &= \cancel{\cos\omega_n a/c}(C\cos\omega t + D\sin\omega t) = 0
\end{aligned}$$

[5]

(d). Now the solution at $x = 0$ must match the given unsteady function $p_0 \sin\Omega t$. We write the generic solution as a sum of all possible solutions:

$$p(0,t) = p_0\sin\omega t = \sum_{n=0}^{\infty} \cos(\omega_n x/a)(C\cos\omega_n t + D\sin\omega_n t)$$

which still obey the initial boundary conditions.

We can multiply both sides of the generic solution at $x = 0$ them by sines or cosines of $\omega_m t$ and integrate over a time T:

$$\int_0^T p_0 \sin \Omega t \cos \omega_m t \; dt = \int_0^T \sum_{n=0}^{\infty} \underbrace{\cos(\omega_n 0/a)}_{1} (C_n \cos \omega_n t \cos \omega_m t + D_n \sin \omega_n t \cos \omega_m t) \; dt$$
$$\int_0^T p_0 \sin \Omega t \sin \omega_m t \; dt = \int_0^T \sum_{n=0}^{\infty} \underbrace{\cos(\omega_n 0/a)}_{1} (C_n \cos \omega_n t \sin \omega_m t + D_n \sin \omega_n t \sin \omega_m t) \; dt$$

It can be shown that (from Maths datebook, Fourier decomposition)

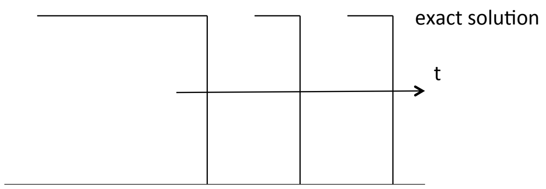
$$\int_0^T \sum_{n=0}^{\infty} \cos \frac{2\pi p t}{T} \cos \frac{2\pi q t}{T} \; dt = \frac{1}{2} \delta_{pq}$$
$$\int_0^T \sum_{n=0}^{\infty} \sin \frac{2\pi p t}{T} \sin \frac{2\pi q t}{T} \; dt = \frac{1}{2} \delta_{pq}$$

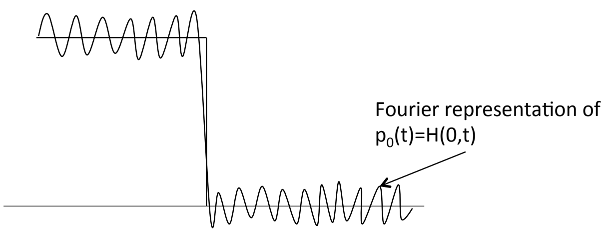
so that if we make $p = 2n + 1$, $q = 2m + 1$, $\frac{c}{a}\frac{\pi}{2} = \frac{2\pi}{T}$, or $T = 4a/c$, this corresponds to the Fourier decomposition of $p_0(t)$, and

$$C_n = \frac{c}{2a} \int_0^{4a/c} p_0(t) \cos(2n+1) \frac{\pi}{2} \frac{c}{a} t \; dt$$
$$D_n = \frac{c}{2a} \int_0^{4a/c} p_0(t) \sin(2n+1) \frac{\pi}{2} \frac{c}{a} t \; dt$$

[7]

(e) For a step function at the origin, the step wave will propagate unchanged at the speed of sound through the tube, and reflect at the end, where there is a pressure node. The solution can be approximated by using the same method in (d), where the step function is represented as the sum of harmonics in the corresponding series solution. [4]


exact solution
t


Fourier representation of
$p_0(t)=H(0,t)$

Section B

4. *QR decomposition and Least-Squares Solution*

(a) $\mathbf{A}$ is a 3×2 matrix, which can be decomposed into $\mathbf{A} = \mathbf{Q} \cdot \mathbf{R}$. where $\mathbf{Q}$ is a 3×3 matrix and $\mathbf{R}$ a 3×2 upper triangular matrix (where the bottom (3-2) rows of $\mathbf{R}$ are zero. Thus

$$\mathbf{R} \begin{bmatrix} c \\ d \end{bmatrix} = \mathbf{Q}^{\mathrm{T}} \mathbf{v}$$

As $\mathbf{R}$ is upper-triangular it is then straight-forward to solve by substitution. [3]

(b) The first column is simply the normalised version of the first column of $\mathbf{A}$ and $r_{11} = 3$

$$\mathbf{q}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Projecting the second column and subtracting yields

$$\mathbf{q}_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

The final column is required to be orthogonal to the other two. Thus

$$\mathbf{Q} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}; \quad \mathbf{R} = \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

[8]

(c) The problem can be solved by obtaining the sum squares of the residuals, which leads to: $\mathbf{Rc} = \mathbf{Q}^{\mathrm{T}}\mathbf{v}$. Solving yields (ignoring last column)

$$\begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -11 \\ 8 \end{bmatrix}$$

so that

$$c = -19/9; d = 4/3$$

[6]

(d) For the squared error to be zero, then the last row of $(\mathbf{q}^{\mathrm{T}})_3 \mathbf{v} = 0$. For a solution

$$\begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2x - 2y + z = 0$$

Thus any point lying on the plane will have zero squared error. [8]

5. *Sum of random variables and moment generating functions*

(a): differentiating the expression yields

$$g'(s) = \lambda \exp(-\lambda(1-s))$$

Thus mean is λ. Differentiating again yields

$$\lambda^2 \exp(-\lambda(1-s))$$

Thus the variance is

$$\sigma^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

[5]

(b): the general form is to convolve the two distributions. In this case one is discrete the other continuous. The form must be

$$p_z(Z) = \sum_{X=0}^{\infty} p_x(X) p_y(Z-X)$$

Thus the distribution is continuous [4]

(c)(i): The MGF can be written

$$\begin{aligned} g_y(s) &= \int p_y(x) \exp(-sx) dx \\ &= \int \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x^2 - 2x\mu + \mu^2 + 2sx\sigma^2)\right) dx \\ &= \int \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}((x - (\mu - s\sigma^2))^2 - s^2\sigma^4 + 2s\sigma^2\mu)\right) dx \\ &= \exp\left(s^2\sigma^2/2 - s\mu)\right) \end{aligned}$$

[7]

(c)(ii) Differentiating this expression yields

$$g_z'(s) = (s\sigma^2 - \mu - \lambda \exp(-s)) \exp\left(s^2\sigma^2/2 - s\mu + \lambda(\exp(-s) - 1)\right)$$

Thus the mean is

$$\mu_z = -g_z'(0) = \mu + \lambda$$

as expected. The second differential is

$$\begin{aligned} g_z''(s) &= (\sigma^2 + \lambda \exp(-s)) \exp\left(s^2\sigma^2/2 - s\mu + \lambda(\exp(-s) - 1)\right) \\ &\quad + (s\sigma^2 - \mu - \lambda \exp(-s))^2 \exp\left(s^2\sigma^2/2 - s\mu + \lambda(\exp(-s) - 1)\right) \end{aligned}$$

equating to zero yields and subtracting the μ_z^2

$$\sigma^2 + \lambda + (\mu + \lambda)^2 - (\mu + \lambda)^2 = \sigma^2 + \lambda$$

These are the values of adding two independent variables. [4]

(d): an error occurs when the magnitude of the noise for any integer is greater than 0.5. For a single integer this can be written as (using the symmetry)

$$2\Phi(-0.5/\sigma)$$

The exception to this is that zero integer value is alway correctly recogised when the received signal is negative (it will always be the closest). Thus the overall expression is

$$\begin{aligned} P_{\mathsf{e}} &= p_x(0)\Phi(-0.5/\sigma) + 2\sum_{x=1}^{\infty} p_x(X)\Phi(-0.5/\sigma) \\ &= (2 - p_x(0))\Phi(-0.5/\sigma) \\ &= (2 - \exp(-\lambda))\Phi(-0.5/\sigma) \end{aligned}$$

[5]

6. *LU Decomposition and Matrix Sub-Spaces*

(a) Performing LU decomposition yields $z = 2$ here

$$\begin{bmatrix} 2 & 4 & 1 & 0 \\ 4 & 3 & z & 5 \\ 2 & 5 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 & 0 \\ 0 & -5 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[8]

(b) To find the general solution find the solution - solve

$$\mathbf{Ly} = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$$

Thus

$$\mathbf{y} = \begin{bmatrix} -4 \\ 10 \\ 0 \end{bmatrix}$$

Then solve

$$\mathbf{Ux} = \mathbf{y} = \begin{bmatrix} -4 \\ 10 \\ 0 \end{bmatrix}$$

Thus setting $x_3 = 0$ and $x_4 = 0$

$$\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

To compute the null-space

$$\begin{bmatrix} 2 & 4 & 1 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence one axis is

$$\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

And the second basis

$$\begin{bmatrix} 2 & 4 & 1 & 0 \\ 0 & -5 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence the general solution is

$$\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}$$

[10]

(c) The left null-space is perpendicular to the column-space - simply cross-product two columns

$$\begin{bmatrix} 7 \\ -1 \\ -5 \end{bmatrix}$$

[4]

(d) The solution would change as there would be no left-null-space, and there would only be a single null-space. [3]