EGT0
ENGINEERING TRIPOS PART IA

Tuesday 11 June 2019 9 to 12.10

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Section A: multiple choice supplementary booklet
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

Questions 1–8: see multiple choice supplementary booklet.

SECTION B

9 (**long**)

(a) Find the general solution to

$$\frac{d^2x}{dt^2} + (2+\alpha)\frac{dx}{dt} + 2\alpha x = f(t), \qquad (1)$$

where $f(t) = 0$ and $\alpha = 3$. [10]

(b) Find the general solution to equation (1) for any α, subject to $f(t) = 0$ and $x(0) = 1$. [10]

(c) Find the general solution to equation (1) for $\alpha = 2$, subject to $f(t) = 1 + t$ and $x(0) = 0$. [10]

10 (**long**) Consider the two coupled difference equations

$$\begin{aligned} x_{n+1} &= \alpha x_n + \tfrac{1}{2} y_n \\ y_{n+1} &= (1-\alpha) x_n + \tfrac{1}{2} y_n, \end{aligned}$$

where $0 \leq \alpha \leq 1$ is a fixed parameter.

(a) Show that the difference equation can be written in the form $\mathbf{z}_{n+1} = A\mathbf{z}_n$, where $\mathbf{z}_n = [\ x_n\ y_n\ ]^t$. State the matrix A. [6]

(b) If $\mathbf{z}_0 = [\ 1\ 0\ ]^t$, compute $\mathbf{z}_n$ as $n \to \infty$. You may use the fact that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = \alpha - \frac{1}{2}$. [12]

(c) Now assume that $\mathbf{z}_0$ is a unit vector and we observe that $\mathbf{z}_2 = [\ 0\ 0\ ]^t$. Calculate α. [12]

SECTION C

11 (**long**) The step response of a linear system is given by

$$\begin{cases} 0 & \text{for } t < 0 \\ A + Be^{-t} + Ce^{-2t} & \text{for } t \geq 0 \end{cases}$$

with constants A, B, and C.

(a) Explain why the system must be second order. Hence, show that its impulse response is given by

$$g(t) = \begin{cases} 0 & \text{for } t < 0 \\ D(e^{-t} - e^{-2t}) & \text{for } t \geq 0 \end{cases}$$

where $D = 2C$. [10]

(b) Find the system response $y(t)$ to the input

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-t} & \text{for } t \geq 0 \end{cases}$$

for all t. [10]

(c) For $T > 0$, find the system response $y(t)$ to the input

$$f(t) = \begin{cases} e^{-t} & \text{for } T \leq t \leq 2T \\ 0 & \text{otherwise} \end{cases}$$

for all t. [10]

12 (**long**) Consider the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = f(t),$$

with

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ 6e^{-t}\cos(t) & \text{for } t \geq 0 \end{cases}$$

and initial conditions $y(0) = y'(0) = 0$.

(a) Find $\bar{y}(s)$, the Laplace transform of $y(t)$. [10]

(b) Write $y(t)$ as a convolution between $f(t)$ and another function $g(t)$. State $g(t)$. [10]

(c) By taking the inverse Laplace transform of $\bar{y}(s)$, find $y(t)$. [10]

END OF PAPER

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Part IA 2019

Paper 4: Mathematical Methods

Sections B and C: Numerical Answers

9. (a) $x(t) = Ae^{-2t} + Be^{-3t}$, for $A, B \in \mathbb{R}$

   (b) when $\alpha \neq 2$: $x(t) = Ae^{-2t} + (1-A)e^{-\alpha t}$, for $A \in \mathbb{R}$
   when $\alpha = 2$: $x(t) = (At+1)e^{-2t}$, for $A \in \mathbb{R}$

   (c) $x(t) = Ate^{-2t} + t/4$, for $A \in \mathbb{R}$

10. (a) $A = \begin{bmatrix} \alpha & \frac{1}{2} \\ 1-\alpha & \frac{1}{2} \end{bmatrix}$

    (b) $\mathbf{z}_n \to \dfrac{1}{3-2\alpha}[1\ 2(1-\alpha)]^t$

    (c) $\alpha = \frac{1}{2}$

11. (b) $y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ D(te^{-t} - e^{-t} + e^{-2t}) & \text{for } t > 0 \end{cases}$

    (c) $y(t) = \begin{cases} 0 & \text{for } t < T \\ De^{-t}(t-T) - De^{-2t}(e^t - e^T) & \text{for } T \leq t \leq 2T \\ DTe^{-t} - De^{-2t}(e^{2T} - e^T) & \text{for } t > 2T \end{cases}$

12. (a) $\bar{y}(s) = \dfrac{6(s+1)}{\left((s+1)^2+1\right)^2}$

    (b) $g(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ e^{-t}\sin(t) & \text{for } t > 0 \end{cases}$

    (c) $y(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 3te^{-t}\sin(t) & \text{for } t > 0 \end{cases}$