EGT1
ENGINEERING TRIPOS PART IB

Tuesday 11 June 2019 2 to 4:10

Paper 1

MECHANICAL ENGINEERING

Answer not more than **four** *questions.*

Answer not more than **two** *questions from each section. All questions carry the same number of marks.*

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately. Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

Answer not more than ***two*** *questions from this section*

1 A famous tower is leaning at an angle of $\theta = 4\,\text{degrees}$ from vertical. It has a total mass m, and can be modelled as a uniform solid cylinder of radius r and height h. At time $t = 0$ the foundations of the tower give way such that it starts to fall, pivoting freely about the centre of the base, which does not move.

(a) Find an expression for the angular velocity of the tower during its fall as a function of the angle of the tower from vertical θ, making a suitable assumption about the effect of the initial lean angle. [8]

(b) Find an expression for the angular acceleration of the tower as a function of θ. [5]

(c) (i) During the fall, at what distance x, as measured from the top of the tower, does the maximum bending moment occur? [10]

(ii) Show that $x = 2h/3$ for the case $r = 0$ (i.e. a thin rod). [2]

2 Figure 1 shows a diagram (not to scale) for a partially complete design of a lifting mechanism that is made from three light rigid bodies AB, CD and BCE. At the instant shown, the coordinates of the points are A = (0,0) mm, B = (0,60) mm, C = (20,60) mm, D = (x,0) mm and E = (110,90) mm.

(a) The distance x is to be chosen, which determines the length of the link CD.

(i) Identify the coordinates of the instantaneous centre of the completed mechanism such that point E instantaneously moves vertically. [4]

(ii) What is the corresponding distance x that results in E moving vertically? [6]

(b) For the case $x = 40$ mm:

(i) If point E has an absolute speed of $5\,\mathrm{mm\,s^{-1}}$ and AB rotates clockwise, find the angular velocity of each component. [10]

(ii) A frictional torque of $2\,\mathrm{N\,mm}$ opposes the motion at each joint A, B, C and D, and a load of $0.1\,\mathrm{N}$ pulls vertically downwards at E. Find the required drive torque at A for the same motion as described in part (b)(i). [5]

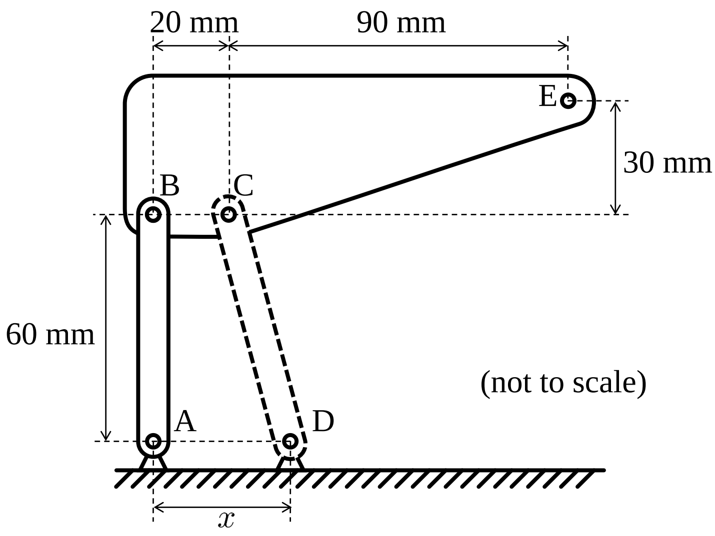


Fig. 1

3 A schematic diagram of a go-kart travelling at velocity $V_0\mathbf{j}$ is shown in Fig. 2(a). The centre of mass G is at a height h above the road, and point A is the centre of the front-right wheel. The unit vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ are fixed to the ground, and $\mathbf{e_r},\mathbf{e}_\theta$ are defined relative to the kart, with $\mathbf{e_r}$ parallel to the wheel axes.

Figure 2(b) shows a view from the side of the kart of the front-right wheel with radius r. Point C defines a general position on the tyre at an angle ϕ from the point of contact with the road, and unit vectors $\mathbf{e_1},\mathbf{e_2}$ rotate with the wheel such that $\mathbf{e_1}$ is in the direction AC. The angular velocity of the kart about a vertical axis is defined to be $\Omega\mathbf{k}$.

(a) Initially, the kart is travelling along a straight path at a constant velocity $V_0\mathbf{j}$ (i.e. $\Omega = 0$) with wheels of radius r rotating at angular speed ω such that there is no slip.

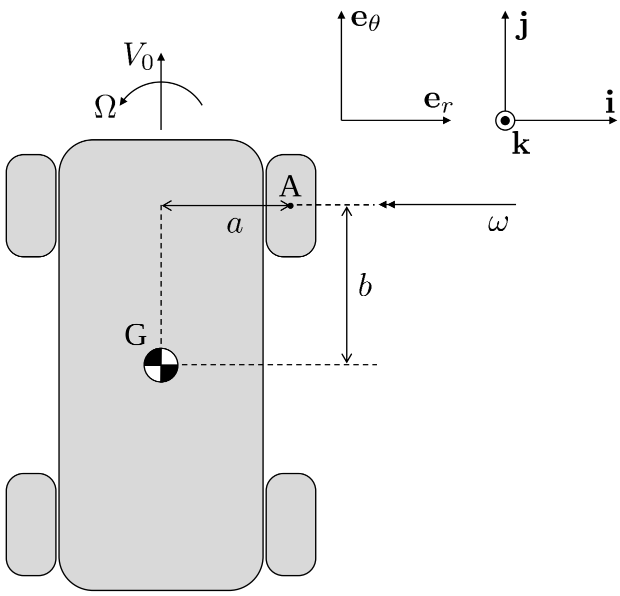
(i) Write down an expression for the position vector of point C relative to G in terms of the unit vectors and parameters defined in Fig. 2. [4]

(ii) Derive expressions for the velocity and acceleration of point C relative to G in terms of the unit vectors and parameters defined in Fig. 2. [6]

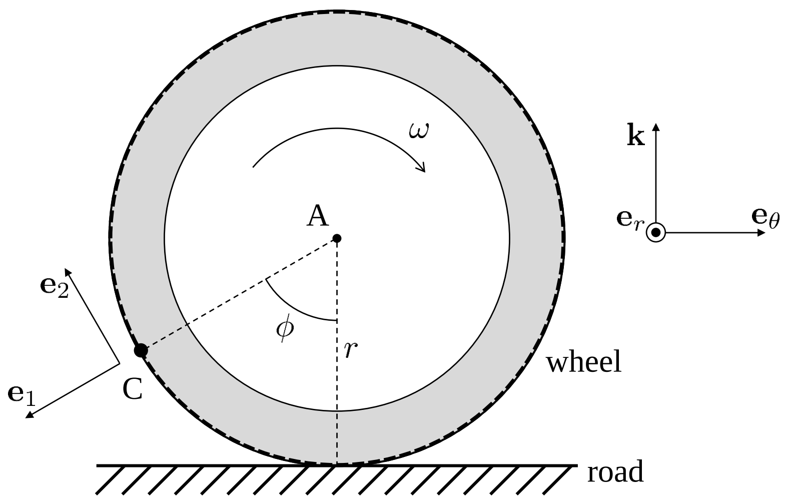
(b) The kart then goes into a spin about a vertical axis. At the instant shown in Fig. 2(a), the angular velocity of the kart is $\Omega\mathbf{k}$ where Ω is constant, the wheels continue to rotate at constant angular speed ω, and the centre of mass of the kart continues to move at constant velocity $V_0\mathbf{j}$. It can be assumed that the driver does not steer the wheels.

(i) Derive expressions for the derivatives $\dot{\mathbf{e}}_r$, $\dot{\mathbf{e}}_\theta$, $\dot{\mathbf{e}}_1$ and $\dot{\mathbf{e}}_2$. [6]

(ii) When C is instantaneously in contact with the ground, find the velocity and acceleration of point C relative to the ground in terms of the unit vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ and $\mathbf{e}_r,\mathbf{e}_\theta$, and the parameters defined in Fig. 2. [9]



(a) Top view of the go-kart



(b) Side view of the front right wheel

Fig. 2

SECTION B

*Answer not more than **two** questions from this section.*

4 A bead of mass m is moving under gravity and slides freely on a rod of length l that is rigidly attached to the top of a pole at a fixed angle θ, as shown in Fig. 3. The distance of the bead from the top of the pole is z. The rod-and-pole assembly is rotating around the vertical axis with fixed angular velocity ω.

(a) Taking z as the generalised coordinate, write down the Lagrangian for this system, and hence derive the equation of motion for z. [10]

(b) Identify the value of z corresponding to the equilibrium position, and determine whether it is stable or unstable. If $\theta = 45^\circ$, $l = \sqrt{2}$ meters, $\omega = \sqrt{10}$ radians/second, and initially $z = l$, describe the subsequent motion of the bead. [7]

(c) Using the first integral of the equation of motion, or otherwise, draw the phase portrait of the system using the displacement z and velocity $\dot{z}$ as axes. [8]

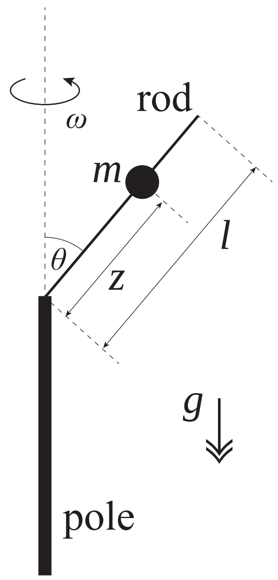


Fig. 3

5 A particle of mass m can slide without friction on a circular arc of radius r, as shown in Fig. 4. The arc is accelerating from rest with constant horizontal acceleration a, and it is in a vertical plane so the effect of gravity should be included.

(a) Show that the Lagrangian L for the system can be written:

$$L = \frac{1}{2}m(r\dot{\theta}\cos\theta - at)^2 + \frac{1}{2}mr^2\dot{\theta}^2\sin^2\theta + mrg\cos\theta$$

[12]

(b) Derive the equation of motion, and comment on its form. [7]

(c) The total angle subtended by the arc is ϕ, i.e. if the particle slides beyond $\theta = \pm\phi/2$ then it will fall off the arc. Derive a formula for the maximum acceleration that allows the particle to remain on the arc in stable equilibrium. [6]

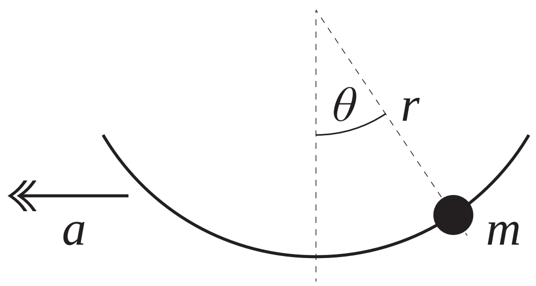


Fig. 4

You might find the following formula useful:

$$a\sin\alpha + b\cos\alpha = \sqrt{a^2+b^2}\sin(\alpha+\beta), \qquad \tan\beta = b/a$$

6 Consider a double pendulum composed of light inextensible rods of length l connecting masses m, as shown in Fig. 5. The angle of the rods from vertical are denoted by θ_1 and θ_2.

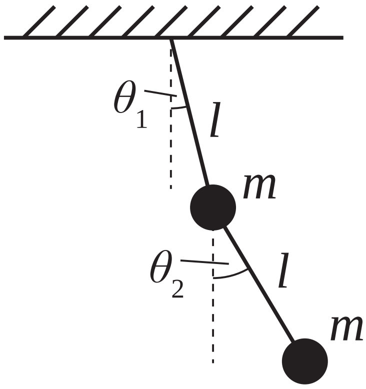


Fig. 5

(a) Derive the Lagrangian L for the system. [8]

(b) Identify all equilibrium states, and determine for each equilibrium whether the normal modes are stable or unstable. [6]

(c) Identify the equilibrium state that has the *second lowest* potential energy, and show that the Lagrangian expanded around that state, using small deviations $\delta\theta_1, \delta\theta_2$, is

$$L = \frac{1}{2}ml^2[\delta\dot{\theta}_1 \;\; \delta\dot{\theta}_2] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \delta\dot{\theta}_1 \\ \delta\dot{\theta}_2 \end{bmatrix} + \frac{1}{2}mgl[\delta\theta_1 \;\; \delta\theta_2] \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\theta_1 \\ \delta\theta_2 \end{bmatrix}$$

[6]

(d) Hence calculate the normal mode frequencies corresponding to the equilibrium state in part (c) and comment on the result. [5]

END OF PAPER

Answers

Q1 (a) $\dot{\theta} = \sqrt{\dfrac{12gh(1-\cos\theta)}{3r^2+4h^2}}$

(b) $\ddot{\theta} = \dfrac{6gh\sin\theta}{3r^2+4h^2}$

(c) $x = \left(\dfrac{h}{3} - \dfrac{r^2}{2h}\right) + \sqrt{\left(\dfrac{h}{3} - \dfrac{r^2}{2h}\right)^2 + \dfrac{r^2}{2}}$

Q2 (a)(i) $(0, 90)$ mm

(ii) $x = 60$ mm

(b)(i) $\omega_{BCE} = -0.0439 \text{ rad s}^{-1}$ ie anticlockwise

$\omega_{AB} = +0.0439 \text{ rad s}^{-1}$ ie clockwise

$\omega_{CD} = +0.0439 \text{ rad s}^{-1}$ ie clockwise

(ii) $T = 23.0$ Nmm

Q3 (a)(i) $\underline{r}_{C/G} = a\underbrace{\underline{i}}_{=\underline{e}_r} + b\underbrace{\underline{j}}_{=\underline{e}_\theta} - r\sin\phi\,\underline{j} - (h - r + r\cos\phi)\underline{k}$

$= a\underline{e}_r + b\underline{e}_\theta + (r-h)\underline{k} + r\underline{e}_1$

(ii) $\dot{\underline{r}}_{C/G} = r\omega\underline{e}_2 = V_0\underline{e}_2$

$= -r\omega\cos\phi\underbrace{\underline{j}}_{=\underline{e}_\theta} + r\omega\sin\phi\,\underline{k}$

$\ddot{\underline{r}}_{C/G} = -r\omega^2\underline{e}_1 = -\dfrac{V_0^2}{r}\underline{e}_1$

$= r\omega^2\sin\phi\underbrace{\underline{j}}_{=\underline{e}_\theta} + r\omega^2\cos\phi\,\underline{k}$

(b) (i) $\dot{\underline{e}}_r = \Omega\underline{e}_\theta = \Omega\underline{j}$

$\dot{\underline{e}}_\theta = -\Omega\underline{e}_r = -\Omega\underline{i}$

$\dot{\underline{e}}_1 = \Omega\sin\phi\,\underline{e}_r + \omega\underline{e}_2$

$= \Omega\sin\phi\,\underline{i} - \omega\cos\phi\underbrace{\underline{e}_\theta}_{=\underline{j}} + \omega\sin\phi\,\underline{k}$

$\dot{\underline{e}}_2 = \Omega\cos\phi\,\underline{e}_r - \omega\underline{e}_1$

$= \Omega\cos\phi\,\underline{i} + \omega\sin\phi\,\underline{e}_\theta + \omega\cos\phi\,\underline{k}$

(3)(b)(ii) $\dot{\underline{r}}_c = (V_0 + a\Omega - r\omega)\underset{\underset{\underline{j}}{\shortparallel}}{\underline{e}_\theta} - b\Omega\,\underline{e}_r$

$\ddot{\underline{r}}_c = (2r\omega\Omega - a\Omega^2)\underset{\underset{\underline{i}}{\shortparallel}}{\underline{e}_r} - b\Omega^2 \underset{\underset{\underline{j}}{\shortparallel}}{\underline{e}_\theta} + r\omega^2 \underline{k}$

Q4 (a) $L = \frac{1}{2}m\left(\dot{z}^2 + \omega^2 z^2 \sin^2\theta\right) - mgz\cos\theta$

$\ddot{z} = \omega^2 z \sin^2\theta - g\cos\theta$

(b) $z_0 = \dfrac{g\cos\theta}{\omega^2 \sin^2\theta}$, unstable

Q5 (b) $\ddot{\theta} = -\frac{a}{r}\cos\theta - \frac{g}{r}\sin\theta$

(c) $a < g\tan\phi/2$

Q6 (a) $L = \frac{1}{2}ml^2\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right] + mgl\left(2\cos\theta_1 + \cos\theta_2\right)$

(b) $\theta_1 = 0,\ \theta_2 = 0$: stable

$\theta_1 = 0,\ \theta_2 = \pi$
$\theta_1 = \pi,\ \theta_2 = 0$
$\theta_1 = \pi,\ \theta_2 = \pi$ | unstable

(c) $\theta_1 = 0,\ \theta_2 = \pi$

(d) $\omega = \sqrt[4]{2}\sqrt{g/l}$