Version TB/4

EGT1
ENGINEERING TRIPOS PART IB

Monday 5 June 2023   9.00 to 11.10

Paper 1

MECHANICAL ENGINEERING

Answer not more than ***four*** *questions.*

Answer not more than ***two*** *questions from each section. All questions carry the same number of marks.*

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

Answer not more than ***two*** *questions from this section*

1 Figure 1 shows a sketch of a windscreen wiper mechanism (not to scale). A drive link OA is fixed at O and rotates anticlockwise at a constant angular velocity ω. A coupling rod AB connects the drive link OA to the windscreen wiper link BC, which forms part of the rigid body BCD. Point C is fixed and CD represents the wiper blade with centre of mass G. Dimensions are shown in the figure. In the position shown, OA is collinear with AB, and the angle ABC is 45°.

(a) Assume that all components are light, including the wiper blade CD:

(i) Identify the location of the instantaneous centre of AB. [3]

(ii) Find the angular velocity of AB and BD. [4]

(iii) Assume that friction between the wiper blade and the windscreen can be modelled as discrete point forces of equal magnitude P that oppose the motion at Points D and G, and that a frictional torque of magnitude Q acts at each joint, O, A, B and C. Find the input power needed to drive the link OA at speed ω. [6]

(b) Now assume that the wiper blade CD is a uniform bar of mass m, that all other links are light, and neglect friction:

(i) Find the acceleration of Point A, $\ddot{\mathbf{r}}_A$, and show that the acceleration of Point B is:

$$\ddot{\mathbf{r}}_B = \frac{5\omega^2 L}{4}(\mathbf{i} - \mathbf{j})$$

[6]

(ii) Find the bending moment at C in BD. [6]

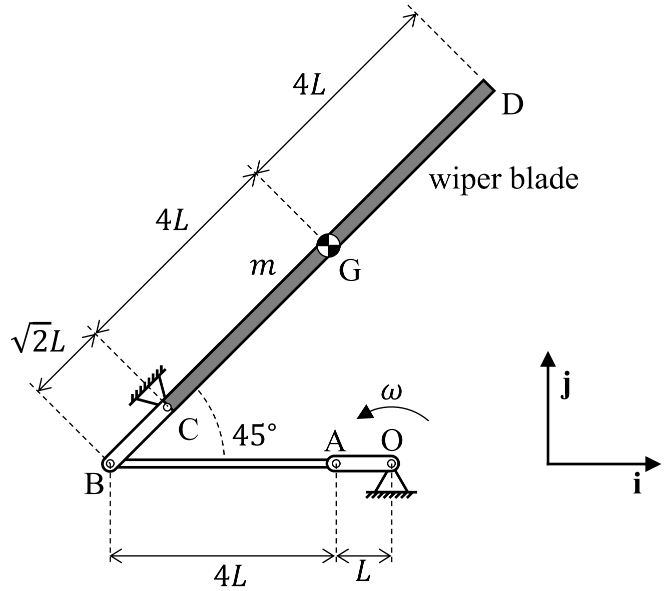


Fig. 1

2 Some flying insects have a pair of organs called 'Halteres' that are used to measure body rotation during flight. A simple illustration of an insect is shown in Fig. 2: (a) shows a three dimensional sketch and (b) shows a rear view. Each Haltere is pivoted at Point O and can be modelled as a mass m on a light inextensible rod of length r oscillating at high frequency such that θ varies sinusoidally, and they remain in a vertical plane orthogonal to $\mathbf{e}_1$. The insect flies in a horizontal plane in the direction of the unit vector $\mathbf{e}_1$. The unit vector $\mathbf{k}$ points vertically upwards, $\mathbf{e}_1^* = \mathbf{k} \times \mathbf{e}_1$, $\mathbf{e}_2$ points in the OA direction, and $\mathbf{e}_2^* = \mathbf{e}_2 \times \mathbf{e}_1$.

(a) For the case when the insect flies along a straight path with velocity $v\mathbf{e}_1$ and acceleration $a\mathbf{e}_1$:

(i) Find expressions for the time derivatives of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_1^*$, $\mathbf{e}_2$, and $\mathbf{e}_2^*$. [4]

(ii) Find the velocity and acceleration vectors $\dot{\mathbf{r}}_\mathrm{A}$ and $\ddot{\mathbf{r}}_\mathrm{A}$ for Point A relative to ground in terms of the given unit vectors. [5]

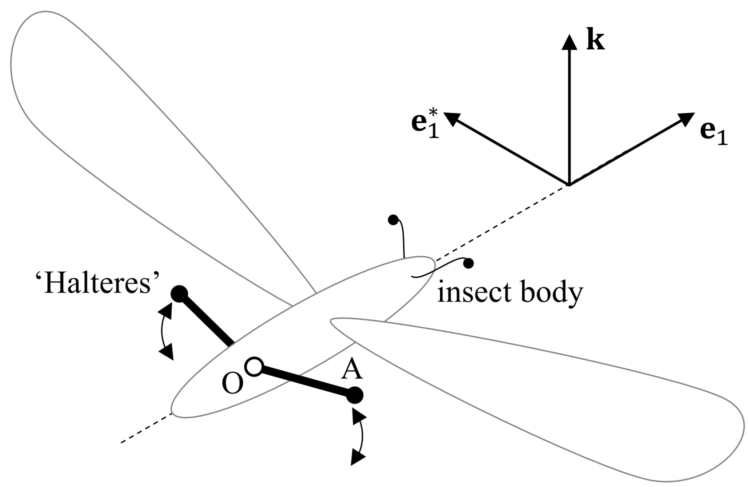
(b) For the case when the insect hovers at a fixed position, such that Point O is stationary, but rotates with angular velocity $\Omega\mathbf{k}$ and angular acceleration $\dot{\Omega}\mathbf{k}$:

(i) Find expressions for the time derivatives of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_1^*$, $\mathbf{e}_2$, and $\mathbf{e}_2^*$. [6]

(ii) Find the velocity and acceleration vectors $\dot{\mathbf{r}}_\mathrm{A}$ and $\ddot{\mathbf{r}}_\mathrm{A}$ for Point A relative to ground in terms of the given unit vectors. [7]

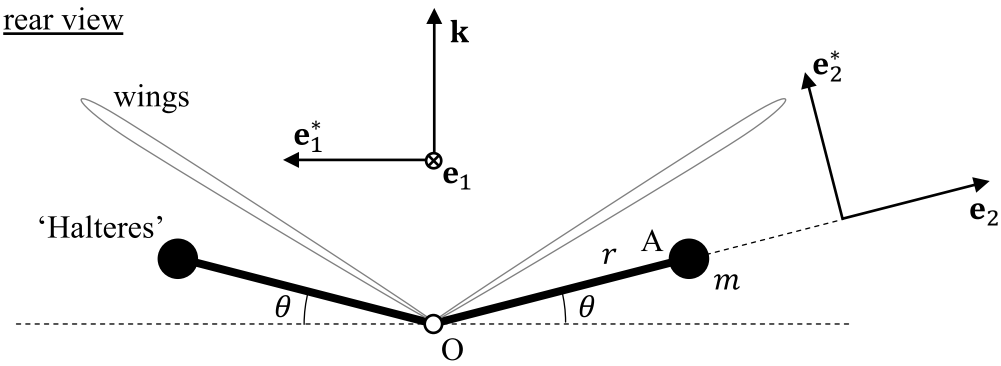
(iii) How could the angular velocity Ω of the insect be identified from the reaction force at Point O? [3]

3D sketch



(a)

rear view



(b)

Fig. 2

3 A hemisphere of mass m and radius R is placed on a horizontal surface as illustrated in Fig. 3. The origin O is defined to be the contact point when the hemisphere is horizontal, and point P is the point of contact at a given instant. The centre of the flat surface is labelled C and the centre of mass G is at a distance $d = 3R/8$ from C along the central axis of the hemisphere. Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ are defined in the figure. The angular displacement from horizontal is denoted by θ, and there is no slip or loss of contact between the hemisphere and the surface.

(a) Find an expression in Cartesian coordinates for the position vector $\mathbf{r}_G$ and velocity vector $\dot{\mathbf{r}}_G$ of the centre of mass. [4]

(b) Derive expressions for the kinetic and potential energy of the hemisphere. Note that the mass moment of inertia of a hemisphere about an axis through the centre of mass, and parallel to the flat face, is given by $I_G = 83mR^2/320$. [6]

(c) Find approximate expressions for the kinetic and potential energy of the hemisphere that are valid for small angles, up to and including quadratic terms involving θ and its derivatives. [4]

(d) If the hemisphere is released from rest at a small angle θ_0, find an expression for $\dot{\theta}^2$ as a function of θ given that the system is conservative, and hence find a second order differential equation relating $\ddot{\theta}$ to θ. [7]

(e) Find an expression for the natural frequency of small amplitude oscillation. [4]

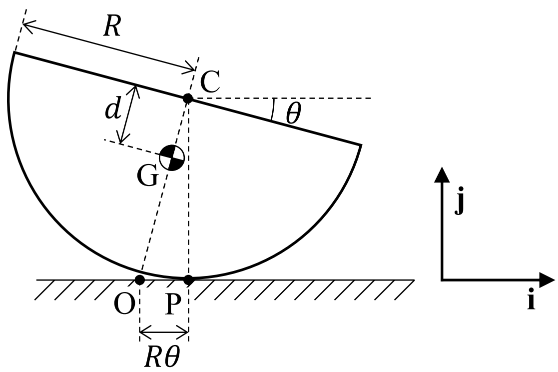


Fig. 3

SECTION B

*Answer not more than **two** questions from this section*

4 A particle of mass m is moving in a potential given by

$$V(x) = kx^4.$$

(a) The particle is held at rest at $x = x_0$ and then released. For the subsequent motion, find the dependence of the period, T, on x_0, k and m. You may use the approximation

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} \approx 1.311.$$

[10]

(b) Draw the phase portrait of the system, using contours corresponding to equal increments of x_0. [7]

(c) The potential is now modified to have the form

$$V_{\text{new}}(x) = -\frac{1}{2}kx^2 + kx^4.$$

Draw the new potential and the new phase portrait. [8]

5 A particle of mass m is moving without friction on a table, and is connected by a light inextensible string to a light spring with spring constant k via a freely rotating pulley as shown in Fig 4. The spring is at its natural length when $r = 0$.

(a) By writing down the Lagrangian, obtain the equations of motion in terms of θ and r, and show that angular momentum is conserved. [5]

(b) Show that the effective potential for the radial degree of freedom is

$$V_{\text{eff}} = \frac{h^2}{2mr^2} + \frac{1}{2}kr^2.$$

[6]

(c) Sketch the effective potential and find the radius and period of circular orbits as a function of the angular momentum h. [7]

(d) By considering small oscillations in radius around the stable circular orbit, obtain the frequency, and hence deduce and draw the shape of deviations from circular orbit. [7]

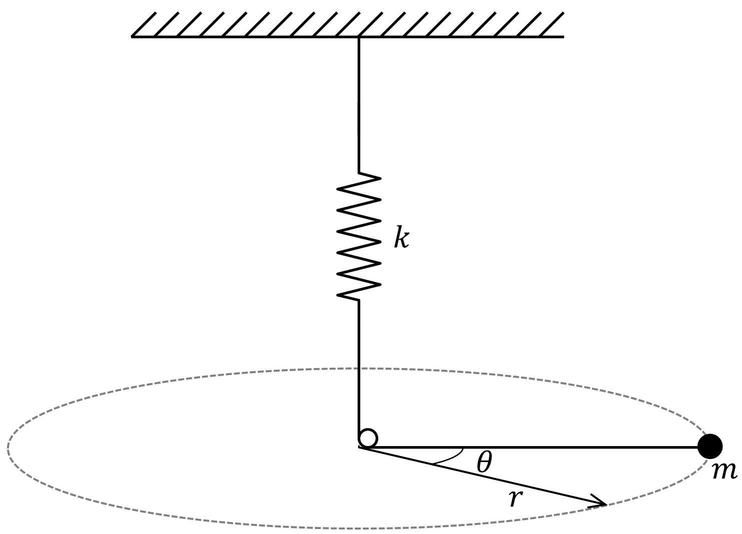


Fig. 4

6 A pendulum consists of a rigid light rod of length l and a bob of mass m, and is attached by a frictionless pin to a mass m that slides without friction supported by two springs, each with constant k, as shown in Fig. 5. With the masses at rest, the springs are at their natural lengths.

(a) Write down the Lagrangian of the system, and use it to derive the equations of motion in the limit of small displacements in matrix form as

$$\mathbf{M}\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = -\mathbf{K}\begin{bmatrix}x\\ \theta\end{bmatrix},$$

and find the matrices $\mathbf{M}$ and $\mathbf{K}$. [10]

(b) Determine the frequencies of small oscillations around equilibrium. [9]

(c) In the limit of large k, find the lowest order correction to the natural frequency of the ordinary pendulum ($\omega_0^2 = g/l$). [6]

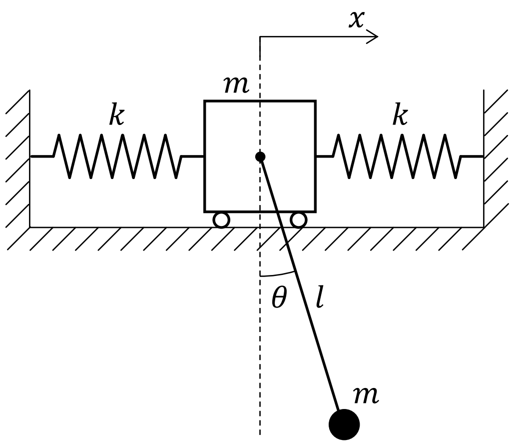


Fig. 5

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IB Mechanics 2023 - Answers

1. (a) i. Instantaneous centre at B
   ii. $\omega_{\mathrm{AB}} = \omega/4$, $\omega_{\mathrm{BD}} = 0$,
   iii. Power$= 5Q\omega/2$

   (b) i. $\ddot{r}_{\mathrm{A}} = \omega^2 L\mathbf{i}$
   ii. $M = 80m\omega^2 L^2/3$
   iii. Power$= 5Q\omega/2$

2. (a) i. $\dot{\mathbf{e}}_1 = \dot{\mathbf{e}}_1^* = 0$
   $\dot{\mathbf{e}}_2 = \dot{\theta}\mathbf{e}_2^*$
   $\dot{\mathbf{e}}_2^* = -\dot{\theta}\mathbf{e}_2$
   ii. $\dot{\mathbf{r}}_{\mathrm{A}} = v\mathbf{e}_1 + r\dot{\theta}\mathbf{e}_2^*$
   $\ddot{\mathbf{r}}_{\mathrm{A}} = a\mathbf{e}_1 + r\ddot{\theta}\mathbf{e}_2^* - r\dot{\theta}^2\mathbf{e}_2$

   (b) i. $\dot{\mathbf{e}}_1 = \Omega\mathbf{e}_1^*$,
   $\dot{\mathbf{e}}_1^* = -\Omega\mathbf{e}_1$
   $\dot{\mathbf{e}}_2 = \Omega\cos\theta\mathbf{e}_1 + \dot{\theta}\mathbf{e}_2^*$
   $\dot{\mathbf{e}}_2^* = -\Omega\sin\theta\mathbf{e}_1 - \dot{\theta}\mathbf{e}_2$
   ii. $\dot{\mathbf{r}}_{\mathrm{A}} = r\Omega\cos\theta\mathbf{e}_1 + r\dot{\theta}\mathbf{e}_2^*$
   $\ddot{\mathbf{r}}_A = \left(r\dot{\Omega}\cos\theta - 2r\Omega\dot{\theta}\sin\theta\right)\mathbf{e}_1 + r\Omega^2\cos\theta\mathbf{e}_1^* - r\dot{\theta}^2\mathbf{e}_2 + r\ddot{\theta}\mathbf{e}_2^*$

3. (a) $\mathbf{r}_{\mathrm{G}} = R\left(\theta - \frac{3}{8}\sin\theta\right)\mathbf{i} + R\left(1 - \frac{3}{8}\cos\theta\right)\mathbf{j}$
   $\dot{\mathbf{r}}_{\mathrm{G}} = R\dot{\theta}\left(1 - \frac{3}{8}\cos\theta\right)\mathbf{i} + \frac{3}{8}R\dot{\theta}\sin\theta\mathbf{j}$

   (b) $T = mR^2\dot{\theta}^2\left(\frac{7}{10} - \frac{3}{8}\cos\theta\right)$
   $V = \frac{3}{8}mgR\left(1 - \cos\theta\right)$

   (c) $V \approx \frac{3}{16}mgR\theta^2$
   $T \approx \frac{13}{40}mR^2\dot{\theta}^2$

   (d) $\dot{\theta}^2 = \frac{15g}{26R}\left(\theta_0^2 - \theta^2\right)$
   $\ddot{\theta} + \frac{15g}{26R}\theta = 0$

(e) $\omega_n^2 = \frac{15g}{26R}$

4. (a) $T = \frac{5.244}{x_0}\sqrt{\frac{m}{2k}}$

5. (a) $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}kr^2$

   $m\ddot{r} + kr - mr\dot{\theta}^2 = 0$

   $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

   (b) /

   (c) $r^4 = \frac{h^2}{mk}$

   $T = 2\pi\sqrt{\frac{m}{k}}$

   (d) $\omega_n = 2\sqrt{\frac{k}{m}}$

6. (a) $L = m\dot{x}^2 + mL\dot{x}\dot{\theta} + \frac{1}{2}mL^2\dot{\theta}^2 - kx^2 - \frac{1}{2}mgL\theta^2$

   $\mathbf{M} = m\,[2 \;\; L \;\; ; \;\; L \;\; L^2]$

   $\mathbf{K} = [2k \;\; 0 \;\; ; \;\; 0 \;\; mgL]$

   (b) $\omega_n^2 = \frac{mg+kL\pm\sqrt{m^2g^2+k^2L^2}}{mL}$

   (c) $\omega_n^2 \approx \frac{g}{L} - \frac{1}{2}\frac{mg^2}{kL^2}$