EGT1
ENGINEERING TRIPOS PART IB

Thursday 6 June 2019   2 to 4.10

Paper 6

INFORMATION ENGINEERING

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Supplementary page: graph template for Question 1 (two copies)

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

*Answer not more than **two** questions from this section.*

1 A car manufacturer adopts a new "climate-control" system, shown in Fig. 1, to regulate the interior air temperature $\theta(t)$ in its vehicles so that it follows the reference temperature $\theta_r(t)$ set by the user. Parameters are $A = 50$, $\alpha = 1$ s, $B = 20$, and $\beta = 100$ s. For now, assume a proportional controller $K(s) = k_p$.

(a) Give rational expressions for the following closed-loop transfer functions, without substituting numerical values for the parameters:

- the transfer function $G_1(s)$ from $\bar{\theta}_r(s)$ to $\bar{\theta}(s)$;
- the transfer function $G_2(s)$ from $\bar{\theta}_r(s)$ to $\bar{e}(s)$. [6]

(b) Assume $k_p = 1$.

(i) Sketch the timecourse of $\theta(t)$ in response to a unit step in θ_r. Do not forget to annotate your graph with any relevant timescales and amplitudes. [3]

(ii) Using the graph template provided at the end of this paper, sketch the Bode diagram of the open loop. Annotate it to show the phase margin. [8]
Please use the graph template attached to the back of this paper, and hand it in with your answer to this question.

(c) Users require that the percentage error in steady-state temperature response to a step change in reference temperature be less than 0.2%, and that the phase margin be at least 40 degrees.

(i) Is this achievable with a simple proportional controller $K(s) = k_p$? If yes, give a value of k_p that works. If no, explain why. [4]

(ii) Someone suggests using a controller of the form $K(s) = 1 + k_d s$. Propose a value of k_d that satisfies the two requirements above, and give a brief explanation. [4]

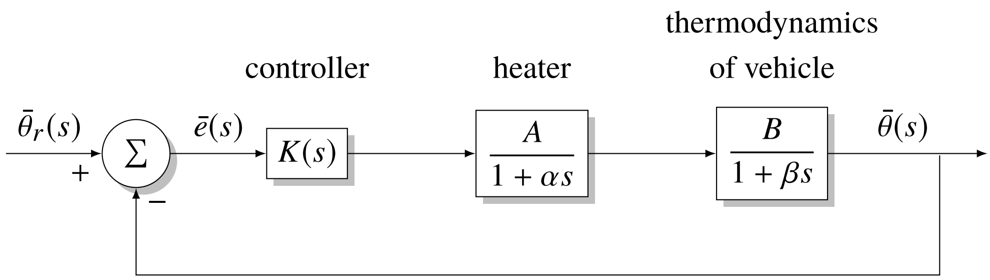

thermodynamics
of vehicle
controller
heater
$\bar{\theta}_r(s)$
$\bar{e}(s)$
Σ
$K(s)$
$\frac{A}{1+\alpha s}$
$\frac{B}{1+\beta s}$
$\bar{\theta}(s)$
+
−

Fig. 1

2 (a) Consider the following open-loop transfer functions:

$$G_1(s) = \frac{\alpha}{1+\tau s} \qquad G_2(s) = \frac{\beta}{1+10\tau s} \qquad G_3(s) = \frac{\gamma}{s(1+\tau s)^2}$$

$$G_4(s) = \frac{\nu}{\tau^2 s^2 + 0.4\tau s + 1} \qquad G_5(s) = \frac{\mu}{\tau^2 s^2 + 2\tau s + 1}$$

where all Greek letters denote unknown constants. Their Nyquist diagrams are shown in Fig. 2, with random labels (a–e) placed on each curve at $\omega = 1$ rad s^{-1}. Match each Nyquist diagram to its corresponding transfer function, reporting your answers in "letter = G_{number}" format. [8]

(b) A stable, second-order linear time-invariant plant is to be regulated in a standard negative feedback configuration, with a proportional controller $K(s) = k_p$ providing input to the plant. A part of the Nyquist diagram of the return ratio is shown in Fig. 3 (solid curve) for $k_p = 1.2$.

(i) Draw a block diagram of the closed-loop system. [3]

(ii) The user requires a phase margin of at least 45 degrees. Give an estimate of the maximum gain k_p that can be used. [5]

(iii) An output disturbance $d(t) = \cos t$ causes the output of the closed-loop system to oscillate. For $k_p = 1.2$, estimate the amplitude of these oscillations at steady state. Give the best upper and lower bounds that you can, given the information provided in Fig. 3. [5]

(iv) The gain is set to $k_p = \frac{6}{7}$, and a delay of τ seconds is introduced in the feedback loop. Give an estimate of the value of τ for which the closed loop becomes unstable. [4]

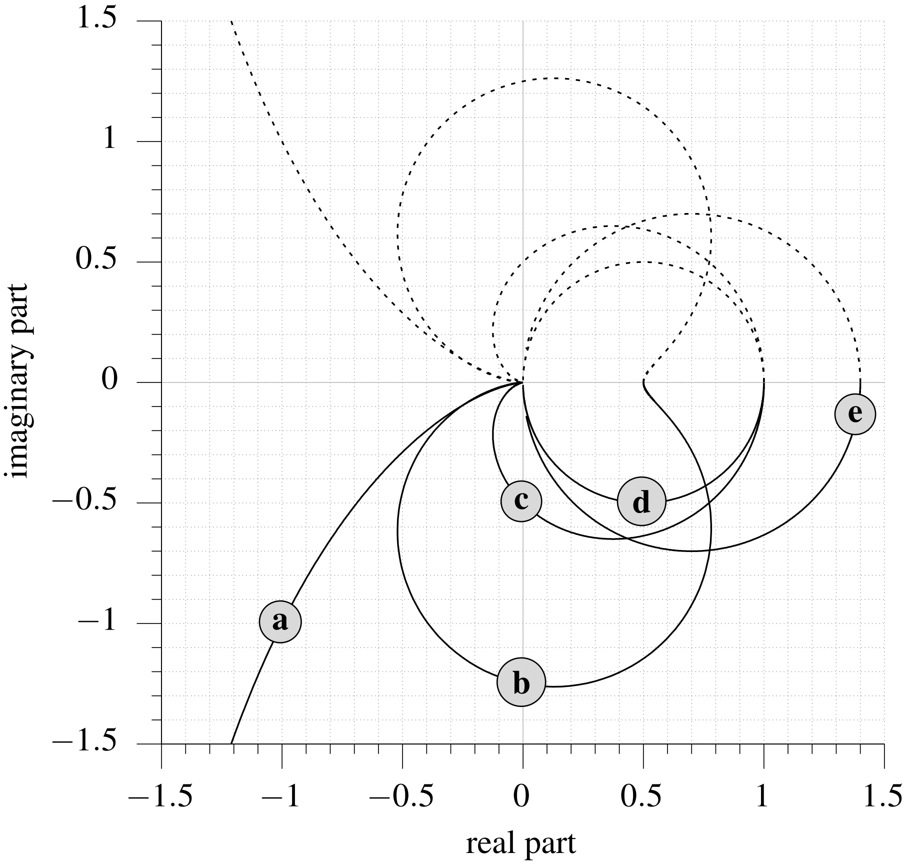

1.5
1
0.5
0
−0.5
−1
−1.5
imaginary part
−1.5
−1
−0.5
0
0.5
1
1.5
real part
a
b
c
d
e

Fig. 2

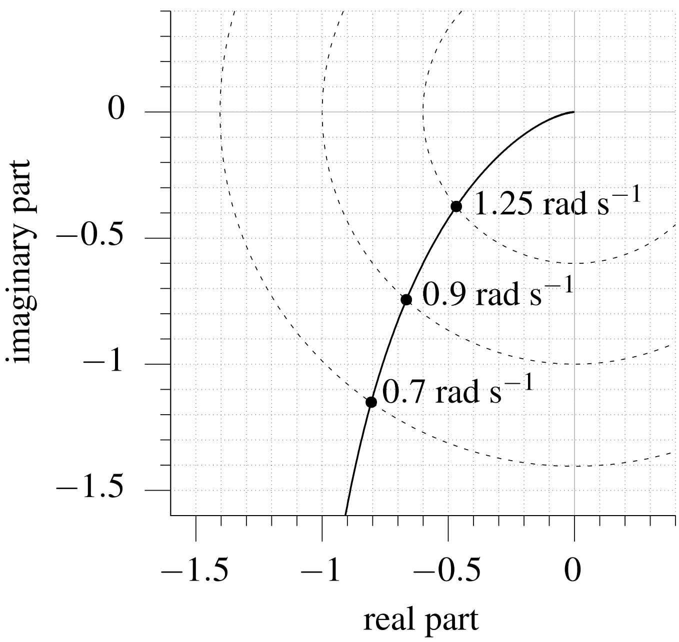

0
−0.5
−1
−1.5
imaginary part
1.25 rad s^{-1}
0.9 rad s^{-1}
0.7 rad s^{-1}
−1.5
−1
−0.5
0
real part

Fig. 3

3 Consider the simplified retinal circuit depicted in Fig. 4 (left). A group of excitatory neurons ('E') are activated by light input (relayed by photoreceptors, not shown). They exert positive feedback onto themselves, and also activate a group of inhibitory neurons ('I'). The inhibitory neurons exert negative feedback onto themselves, and also inhibit the excitatory neurons.

The overall activation of each group of neurons can be modelled as a simple linear system. The interplay between the two groups is summarized in the block diagram shown in Fig. 4 (right). The time constant is set to $\tau = 20$ ms. The goal of this question is to investigate the extent to which this circuit can amplify light input to overcome the detrimental impact of noise in the optic nerve.

(a) Based on the block diagram described in Fig. 4b, derive a pair of coupled differential equations describing the temporal evolution of $x_{\mathrm{E}}(t)$ and $x_{\mathrm{I}}(t)$ respectively. [6]

(b) Set $\beta = 0$, i.e. ignore the entire lower half of the block diagram.

(i) Find the closed-loop transfer function from $\bar{u}(s)$ to $\bar{x}_{\mathrm{E}}(s)$. For what values of α is the closed loop stable? [3]

(ii) Set $\alpha = 0.9$. Sketch the response of the system to a unit step in light intensity $u(t)$. What is the steady-state gain? Approximately how long does $x_{\mathrm{E}}(t)$ take to reach 95% of its steady-state value? [4]

(c) Now take into account the inhibitory feedback loop, and set $\beta = \alpha$.

(i) Find the closed-loop transfer function from $\bar{u}(s)$ to $\bar{x}_{\mathrm{E}}(s)$. For what values of α is the closed loop stable? [5]

(ii) Set $\alpha = \beta = 9$. What is the new steady-state gain? Without calculations, but with justification, give an estimate of the time it takes $x_{\mathrm{E}}(t)$ to reach its steady-state value in response to a step input. [4]

(iii) Give a one-sentence summary of the role of the inhibitory feedback loop in this neural circuit. [3]

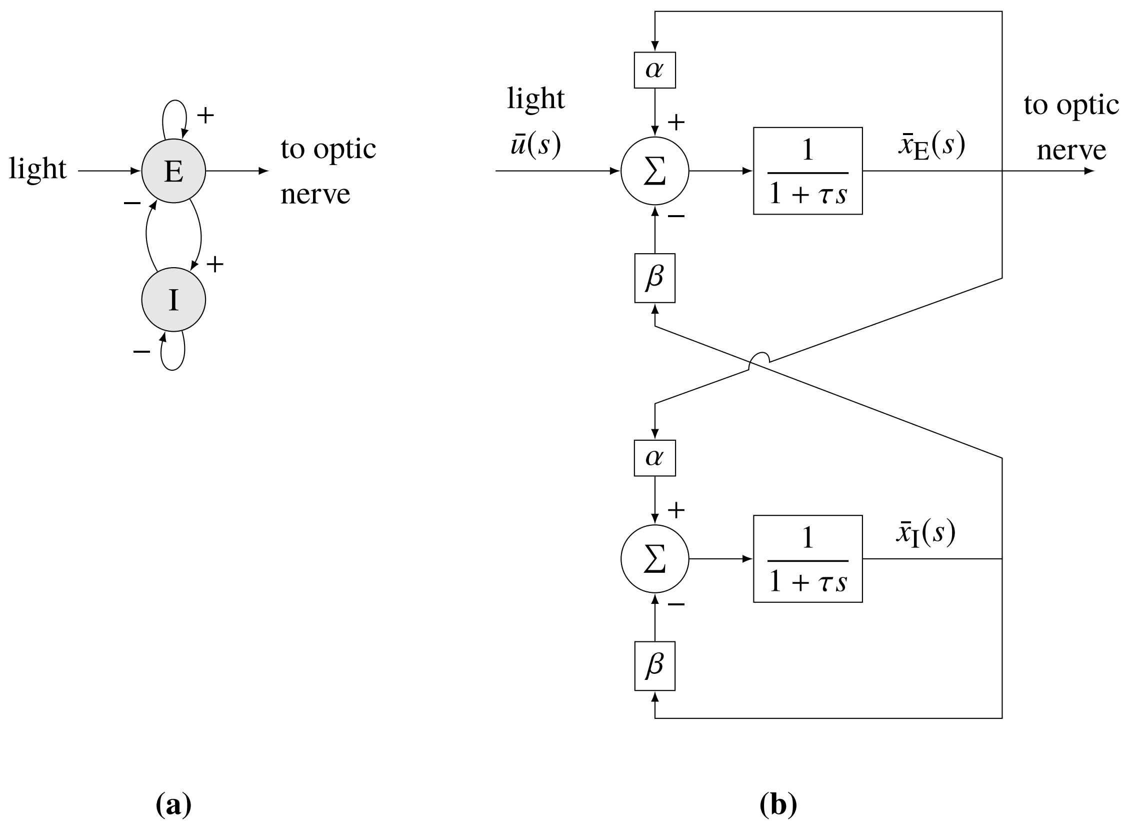

light
E
I
to optic
nerve
+
−
+
−
light
$\bar{u}(s)$
α
β
Σ
$\frac{1}{1+\tau s}$
$\bar{x}_{\mathrm{E}}(s)$
$\bar{x}_{\mathrm{I}}(s)$
to optic
nerve
(a)
(b)

Fig. 4

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) Suppose that the function $f(x)$ has Fourier transform $F(\omega)$. Prove that the Fourier transform of $F(x)$ is $2\pi f(-\omega)$. [5]

(b) Consider the function $g_a(x)$ defined as

$$g_a(x) = \begin{cases} \exp(-ax) & x \geq 0 \\ -\exp(ax) & x < 0 \end{cases},$$

where $a > 0$. Find the Fourier transform, $G_a(\omega)$, of $g_a(x)$. [6]

(c) Find the Fourier transform, $H(\omega)$, of $h(x) = 1/x$, using the duality property from part (a) together with the fact that the functions in part (b) satisfy

$$\lim_{a \to 0} g_a(x) = \text{sign}(x) \qquad \text{and} \qquad \lim_{a \to 0} G_a(\omega) = \frac{2}{j\omega}.$$

[5]

(d) Let $z(x)$ and $y(x)$ have corresponding Fourier transforms $Z(\omega)$ and $Y(\omega)$.

(i) Write the inverse Fourier transform of $Z(\omega - \tau)$ in terms of $z(x)$. [1]

(ii) Using your solution to part (d) (i), or otherwise, derive the inverse Fourier transform, $m(x)$, of the convolution of $Z(\omega)$ and $Y(\omega)$. [3]

(e) Find the Fourier transform, $M(\omega)$, of $m(x) = h(x)k(x)$, where $h(x)$ is given in part (c) and $k(x) = \sin x/x$. You may use your solutions to parts (c) and (d) (ii), together with the fact that the Fourier transform of $k(x)$ is

$$K(\omega) = \begin{cases} \pi & -1 < \omega < 1 \\ 0 & \text{otherwise.} \end{cases}$$

[5]

5 (a) We sample a signal $x(t)$ by the pulse train $\delta_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$. Using the Fourier series representation of $\delta_p(t)$, we can represent the sampled signal $x_s(t)$ as $x_s(t) = \dfrac{1}{T} \sum\limits_{n=-\infty}^{\infty} x(t) e^{jnt2\pi/T}$. Using this result, show that the Fourier transform $X_s(\omega)$ of $x_s(t)$ satisfies

$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n2\pi/T),$$

where $X(\omega)$ is the Fourier transform of $x(t)$. [6]

(b) Consider a finite set of N samples $\{x_0, x_1, \ldots, x_{N-1}\}$ from the signal $x(t)$ in part (a). The discrete Fourier transform (DFT) of this sequence is $\{X_0, X_1, \ldots, X_{N-1}\}$, where

$$X_k = \sum_{n=0}^{N-1} x_n e^{-jkn2\pi/N} \quad \text{for} \quad 0 \geq k \geq N-1\,.$$

Show that, if the samples $\{x_0, x_1, \ldots, x_{N-1}\}$ are real, then $X_{N-k} = X_k^*$ for $k = 0, 1, \ldots, N-1$, where * denotes the complex conjugate. [6]

(c) A three-bit quantiser with a signal range of -1 to $+1$ is used to quantise samples of the triangular wave $x(t)$ shown in Fig. 5 (the wave is periodic with period $T = 4$). Suppose that the quantiser has eight uniformly spaced levels given by $\{(2m-1)/8\}$, for integers $-3 \leq m \leq 4$.

(i) If the amplitude $A = 1$, calculate the mean square quantisation error for the sampled signal $x(0.2k)$ for $k = \ldots, 0, 1, 2, \ldots$. [4]

(ii) If $A = 0.1$, calculate the mean square quantisation error for the sampled signal $x(0.2k)$ for $k = \ldots, 0, 1, 2, \ldots$. [4]

(iii) Express the results of parts (i) and (ii) as signal-to-noise ratios (SNR) in dB, and briefly describe how you can construct an eight-level quantiser that provides good SNR for both small and large signal amplitudes. [5]

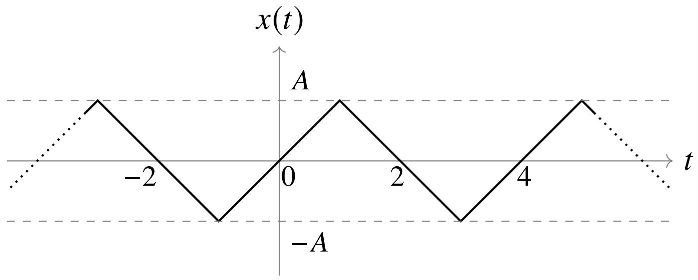


Fig. 5

6 Consider a Pulse Amplitude Modulation (PAM) waveform

$$x(t) = \sum_k X_k p(t - kT),$$

where the information symbols $X_0, X_1, X_2, \ldots$ are drawn from the constellation $\{-3A, -A, A, 3A\}$. The pulse $p(t)$ is as shown in Fig. 6, and $T = 10^{-5}$s.

At the receiver, the demodulator filters the received waveform using a filter with impulse response $p(-t)$, and samples the filter output $r(t)$ at times mT for $m = 0, 1, 2, \ldots$.

(a) What is the rate of transmission of the modulator, in bits per second? [2]

(b) Show that the pulse $p(t)$ has unit energy. [3]

(c) Obtain an expression for the sampled filter output $\{r(mT)\}_{m\geq 0}$, and show that in the absence of noise, $r(mT) = X_m$. [5]

(d) When there is receiver noise, the sampled filter output, denoted by $Y_m = r(mT)$, is given by

$$Y_m = X_m + N_m, \qquad \text{for } m \geq 0,$$

where N_m is independent Gaussian noise with zero mean and variance σ^2. What is the optimum decision rule, assuming that the four constellation symbols are equally likely? [2]

(e) For the decision rule in part (d), compute the probability of detection error for each of the symbols, and hence obtain an expression for the overall probability of detection error. Express your answer in terms of the ratio E_b/σ^2 and the Q-function, where E_b is the average energy per bit of the constellation. Here, $Q(y) = 1 - \Phi(y)$ where $\Phi(y)$ is the Gaussian cumulative distribution function. You may assume that all the constellation symbols are equally likely. [7]

(f) The *power spectral density* of a PAM signal is defined as

$$S_X(f) = \frac{E_s}{T}\,|P(f)|^2 \qquad \text{for } -\infty < f < \infty,$$

where T is the symbol period, E_s is the average energy per symbol, and $P(f)$ is the Fourier transform of the pulse $p(t)$. Compute and sketch the power spectral density of the PAM signal generated with $p(t)$ in Fig. 6, and comment on how fast it decays with growing f. [6]

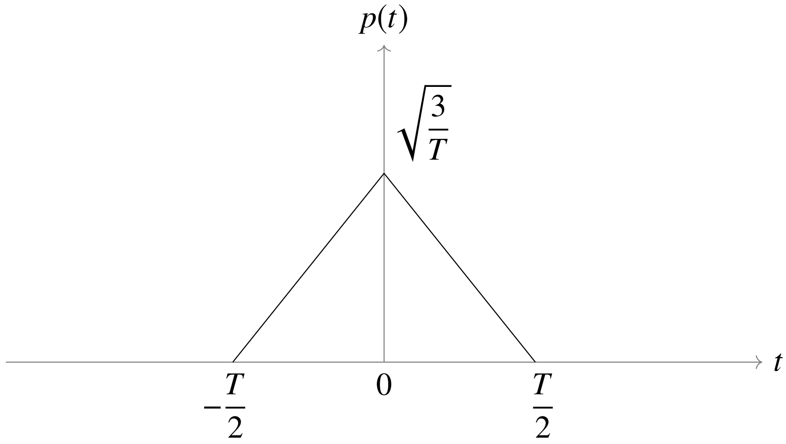


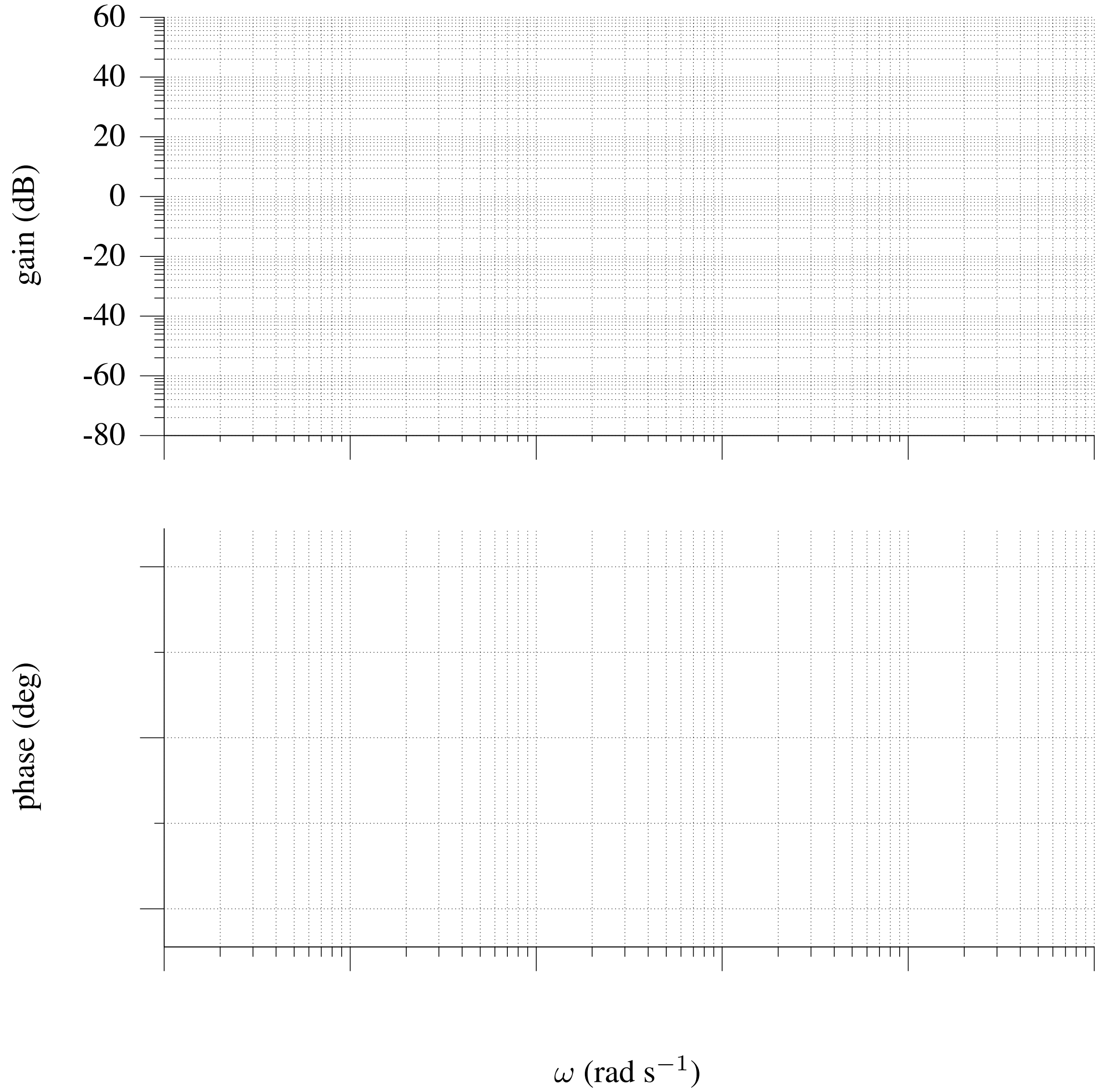
Fig. 6

END OF PAPER

THIS PAGE IS BLANK

Version GH/JMHL/5 Candidate Number:

EGT1
ENGINEERING TRIPOS PART IB
Thursday 6 June 2019, Paper 6, Question 1.



The graph template above is provided for Question 1.(b).(ii). It should be annotated with your constructions and handed in with your answer to this question.