EGT1
ENGINEERING TRIPOS PART IB

Thursday 10 June 2021 13:30 to 15:40

Paper 6

INFORMATION ENGINEERING

*This is an **open-book** exam.*

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the top sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
You have access to the Engineering Data Book, online or as your hard copy.

10 minutes reading time is allowed for this paper at the start of the exam.

The time allowed for scanning/uploading answers is 20 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

SECTION A

*Answer not more than **two** questions from this section.*

1 (a) Consider the following open-loop transfer functions:

$$G_1(s) = \frac{\alpha(1+100s)}{(1+0.2s+s^2)} \qquad G_2(s) = \frac{\beta(1-s)}{(1+20s)} \qquad G_3(s) = \frac{\gamma(1+100s)}{(1+2s+s^2)}$$

$$G_4(s) = \frac{\eta s}{(1+s)(1+100s)} \qquad G_5(s) = \frac{\kappa(1+s)}{(1+20s)}$$

where $\alpha \leq \beta \leq \gamma \leq \eta \leq \kappa$ are constants. Their Bode diagrams are shown in Fig. 1 below, with random labels (a-e) placed on each gain curve, and with random labels (f-k) placed on each phase curve.

(i) Match each gain curve to its corresponding transfer function, reporting your answers in "letter = G_{number}" format. Justify your answer. [4]

(ii) Match each phase curve to its corresponding transfer function, reporting your answers in "letter = G_{number}" format. Justify your answer. [4]

(b) The rest of this question focuses on $G_4(s)$ above, now with $\eta = 10^4$.
Let $\bar{x}(s) = G_4(s)\bar{u}(s)$ in the Laplace domain. Express this relationship as a differential equation in the time domain. [4]

(c) In order to track a desired output $x_d(t)$, G_4 is regulated by the negative feedback loop shown in Fig. 2, involving a proportional controller k_p and a sensor with transfer function $\frac{1}{1+s}$.

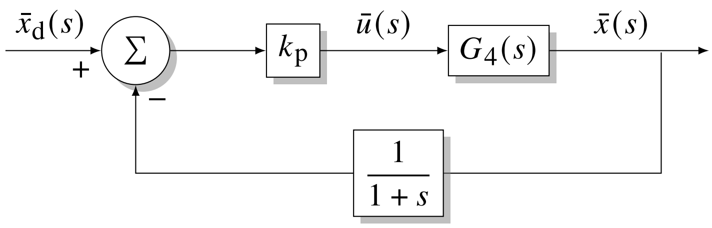

Fig. 2

(i) Set $k_p = 1$. The phase part of the Bode diagram of the return ratio is given in Fig. 3 below. Draw the gain diagram (asymptotes only). [5]

(ii) What is the phase margin? Justify your answer. [4]

(iii) Find the value of k_p that achieves a phase margin of 45 degrees. [4]

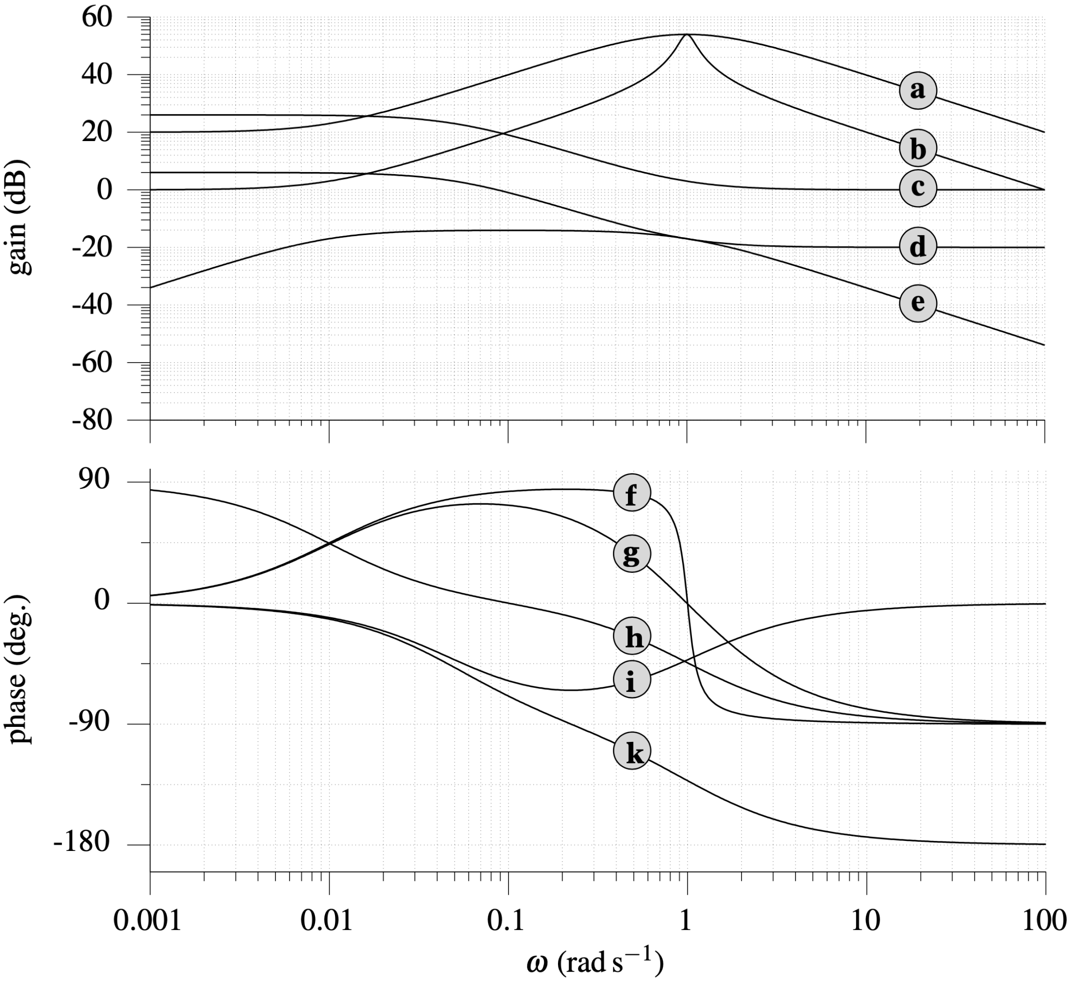
60
40
20
0
-20
-40
-60
-80
gain (dB)
a
b
c
d
e
90
0
-90
-180
phase (deg.)
f
g
h
i
k
0.001
0.01
0.1
1
10
100
ω (rad s^{-1})

Fig. 1

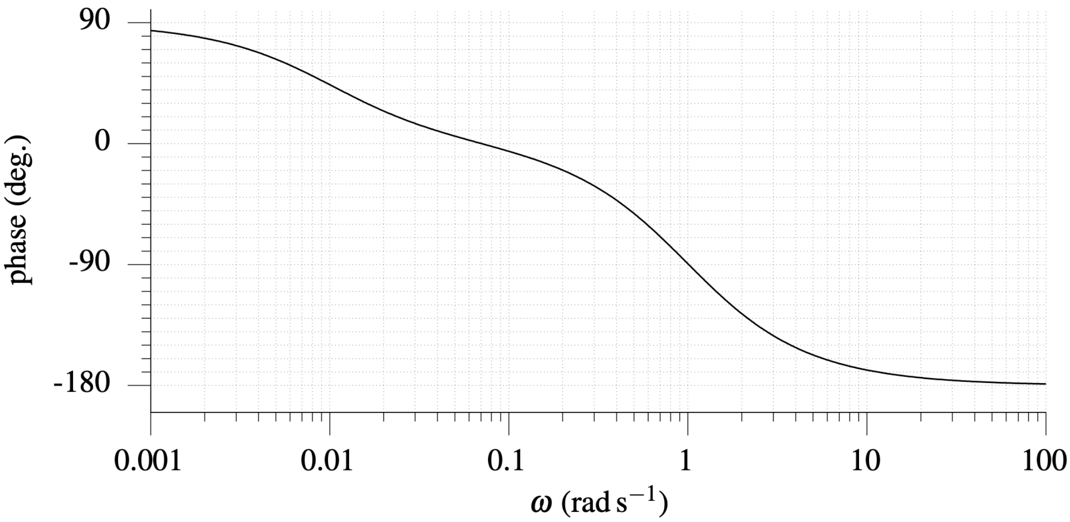
90
0
-90
-180
phase (deg.)
0.001
0.01
0.1
1
10
100
ω (rad s^{-1})

Fig. 3

2 A linear system with transfer function $G(s)$ is controlled in a unity gain negative feedback loop with pre-compensator $K(s)$, as depicted in Fig. 4.

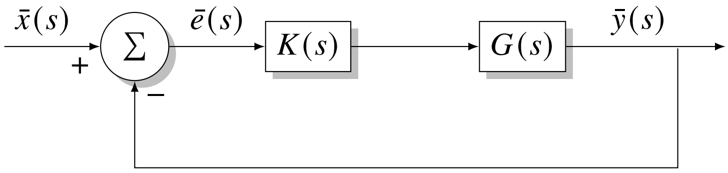

Fig. 4

The Nyquist diagram of the open loop with $K(s) = 1$ is shown as a *solid line* in Fig. 5 below, for the whole range of positive frequencies.

(a) $G(s)$ has no zeros. How many poles does it have? [3]

(b) What is the closed-loop transfer function from $\bar{x}(s)$ to $\bar{e}(s)$? [3]

(c) Consider a simple compensator $K(s) = k_{\mathrm{p}}$. Derive the value of k_{p} that achieves a 2% steady-state error in step response. Show your working out. [5]

(d) The Nyquist diagram of $30G(s)$ is shown in Fig. 5 as a *dashed* line. Use this to estimate the phase margin when k_{p} has the value found in (c). [5]

(e) A proportional integral controller $K(s) = (s + \alpha)/s$ is used instead, where α is some constant. A part of the Nyquist diagram of $K(s)G(s)$ is shown in Fig. 5 as a *dashed-dotted* line for positive frequencies. Here, $K(s)$ is such that the diagram asymptotes onto the y-axis as ω tends to zero.

(i) What can you say about the steady-state error in step response, and why? [4]

(ii) Knowing that $G(j\omega) \approx 1 - 20j\omega$ for small positive ω, derive the value of α. [5]

(cont.

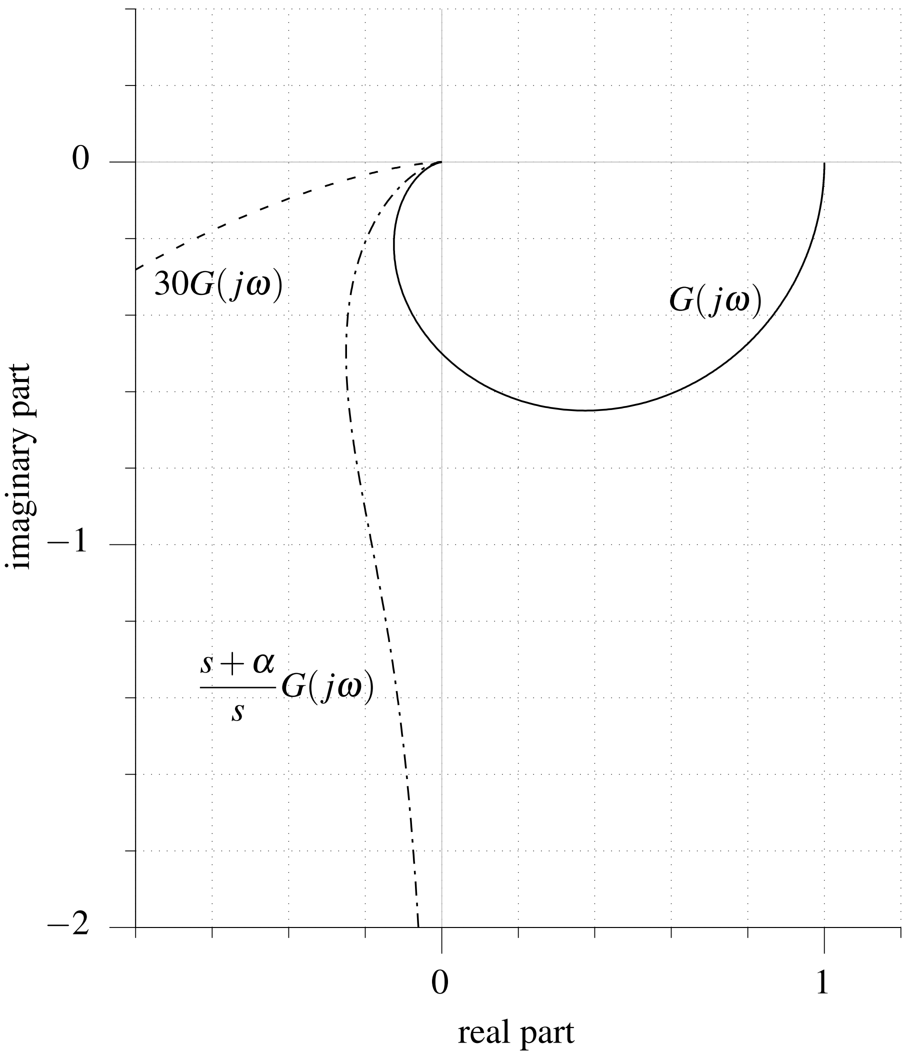
0
−1
−2
imaginary part
0
1
real part
$30G(j\omega)$
$G(j\omega)$
$\frac{s+\alpha}{s}G(j\omega)$

Fig. 5

3 Collaborative robots use elastic joints to improve safety. Consider the following model of a single-link manipulator

$$I\ddot{\theta} + c\dot{\theta} + k(\theta - u) = \tau \qquad (1)$$

where $\theta(t)$ is the angular position of the link, $u(t)$ is the angular position of the driving motor (both measured in radians), and τ is the external torque acting on the link (measured in N m). The motor is connected to the link through a torsional spring with constant $k = 4$ N m rad^{-1}. In the following, $I = 1$ kg m^2 rad^{-1} and $c = 4$ kg m^2 s^{-1} rad^{-1} (damping).

For position control, the motor angle u is set by a proportional derivative controller

$$\bar{u}(s) = \left(k_\mathrm{p} + k_\mathrm{d}s\right)\left[\bar{\theta}_\mathrm{d}(s) - \bar{\theta}(s)\right]$$

where θ_d is the desired link position (reference).

(a) From Eq. 1, derive the open-loop transfer function $G(s)$ from $\bar{u}(s)$ to $\bar{\theta}(s)$ and show that the robot is stable for any $c > 0$ and $k > 0$. [3]

(b) Derive the expression in the time domain of the open-loop response of the robot for a step input u and for $\tau = 0$. [4]

(c) Sketch the open-loop response in (b) and show how this response would change for smaller/larger values of the spring constant k. Justify your answer. [4]

(d) Proceed now to tune the controller gains. Set $k_\mathrm{p} = 100$.

(i) Draw the block diagram of the closed-loop system. [2]

(ii) Select the derivative gain k_d to avoid oscillations in closed loop, by enforcing a damping ratio of 1. Show your working out. [4]

(iii) Show that, in closed loop, the steady state deviation of the output θ to a unit step torque τ is less than 1%. [4]

(e) Suppose that $\tau = -\gamma\dot{\theta} + \tau_\mathrm{ext}$ for some $\gamma > 0$ (additional dissipation). Discuss how this affects the transient behaviour of the closed-loop system and its steady state response to unit step torque τ_ext. [4]

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) Let $f(t)$ have Fourier transform $F(\omega) = 2a/(a^2+\omega^2)$ with $a > 0$. Derive the Fourier transform of $f(\alpha t)$, that is, the time-scaled version of $f(t)$ for $\alpha \neq 0$. [4]

(b) State the convolution theorem for the Fourier transform, and hence (or otherwise) derive the Fourier transform $Z(\omega)$ of the triangular pulse $z(t)$ in Fig. 6. For this, note that $z(t) = \int_{-\infty}^{\infty} y(\tau)y(t-\tau)\, d\tau$, where $y(t)$ is the rectangular pulse also shown in Fig. 6. [4]

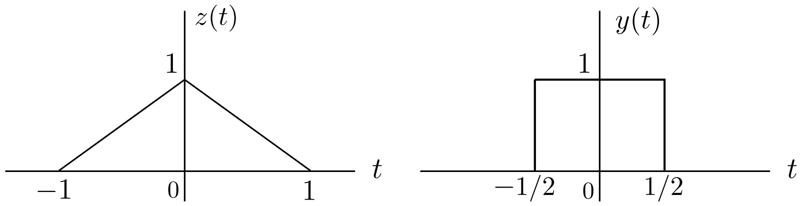

Fig. 6

(c) Consider the signal $x(t)$ depicted below in Fig. 7. Using the results from parts (a) and (b), or otherwise, derive its Fourier transform $X(\omega)$. [4]

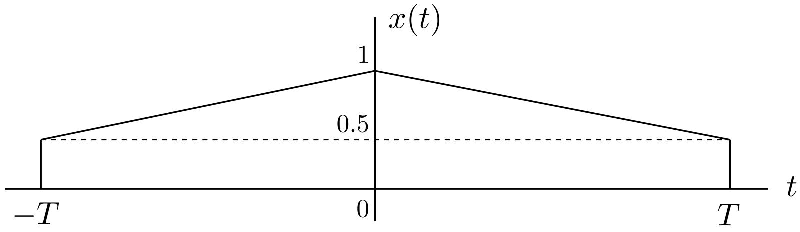

Fig. 7

(d) Given that the Fourier transform of $p(t) = e^{-2|t|}$ is $P(\omega) = 4/(4+\omega^2)$,

(i) use the linearity and duality properties, or otherwise, to derive the Fourier transform of $g(t) = 1/(4+t^2)$, [4]

(ii) use your solution to part (d)(i) and the frequency shift theorem, or otherwise, to derive the Fourier transform of $m(t) = \sin(2t)/(4+t^2)$. [3]

(e) Using Parseval's Theorem and the hints below, or otherwise, derive the energy of the signal $s(t) = \cos(t)\text{sinc}(t)$. [6]

Hints: the Fourier transform of $\text{sinc}(t)$ is a rectangular pulse of width 2 and height π, and the Fourier transform of $\cos(t)$ is $\pi[\delta(\omega+1)+\delta(\omega-1)]$.

5 (a) Consider a train of δ-functions located at intervals of duration T, with the following Fourier series representation:

$$\delta_p(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn2\pi t/T}.$$

This train is used to sample a signal $x(t)$ with Fourier transform $X(\omega) = e^{-2|\omega|}$.

(i) Recall that the Fourier transform of $e^{j\omega_0 t}x(t)$ is $X(\omega - \omega_0)$. Derive the Fourier transform of the sampled signal. Explain if the original signal can be reconstructed from the sampled one. [3]

(ii) Using Parseval's theorem, find the cutoff frequency of an ideal unit gain lowpass filter that keeps 80% of the energy of the original signal. [4]

(iii) Apply the low pass filter from (a)(ii) to $x(t)$ and sample the filtered signal using the train of δ-functions, $\delta_p(t)$. Find the value of T that guarantees perfect reconstruction of the original signal from the sampled filtered signal. [3]

(b) The Discrete Fourier Transform (DFT) of a sampled signal x_n with N samples is

$$X_m = \sum_{n=0}^{N-1} x_n e^{-jnm2\pi/N}, \qquad m = 0, \ldots, N-1.$$

(i) Let $N = 4$ and let the sampling interval be $T = 50$ seconds. Indicate which frequency component of $\{x_0, x_1, x_2, x_3\}$, in radians per second, is measured by X_m. [3]

(ii) Let $X = \{0, 2j, 0, -2j\}$ be the 4 DFT values obtained.
Reconstruct $\{x_0, x_1, x_2, x_3\}$. [5]

(c) The periodic signal $x(t)$ in Fig. 8 is sampled at intervals $n \times 0.25$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$ and quantised using six equally spaced levels given by $\{(2m-1)/6\}$ for integers m between -2 and 3. Calculate the SNR of the quantised signal in dB. [7]

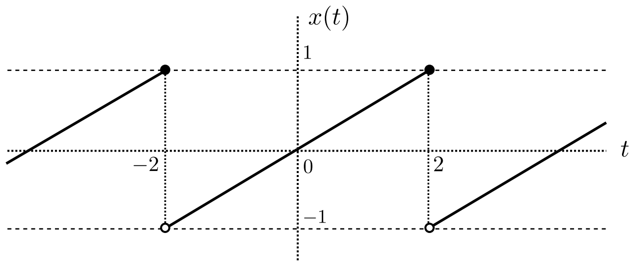

Fig. 8

6 (a) A speech signal $f(t)$ with values in the range ±1 is to be transmitted via modulation on a radio channel centered at 900 kHz. The modulation index is 0.5. The expression for the modulated signal is

$$s(t) = (15 + af(t))\cos(\omega_c t)\,,$$

with ω_c measured in radians per second.

(i) What are the values of the constants a and ω_c? [4]

(ii) Derive the spectrum $S(\omega)$ of the modulated signal in terms of the Fourier transform $F(\omega)$ of the signal $f(t)$. [4]

(iii) What bandwidth should be allocated to this channel if $f(t)$ is a baseband signal with one-sided bandwidth 5 kHz and a gap of 3 kHz must be maintained with the channels on either side of the passband spectrum? [2]

(iv) An alternative modulated signal is given by $u(t) = af(t)\cos(\omega_c t)$. Discuss the advantages/disadvantages of $s(t)$ and $u(t)$ for transmitting $f(t)$. [4]

(b) A sequence of bits is to be transmitted over a channel using Pulse Amplitude Modulation with a binary constellation given by $\{-A, A\}$. If the baseband waveform is formed by a pulse $p(t)$ of duration $T = 10^{-3}$ seconds, what is the transmission rate in bits per second? [2]

(c) For the sequence in Part (b), let X_k denote the k-th constellation symbol being transmitted. The modulated signal is transmitted over an additive Gaussian noise channel and the discrete-time received sequence is $Y_k = X_k + N_k$, where $k = 0, 1, 2, \ldots$ and N_k is Gaussian noise of mean zero and variance σ^2.

(i) The *a priori* symbol probabilities are $p(X_k = A) = \frac{2}{3}$ and $p(X_k = -A) = \frac{1}{3}$. The optimal choice for X_k given Y_k is obtained by maximising the posterior probability $p(X_k|Y_k) = p(Y_k|X_k)p(X_k) \times$ constant. Find the threshold θ such that the receiver chooses $X_k = A$ if $Y_k > \theta$ and $X_k = -A$ otherwise. [5]

(ii) Write down the probability of detection error in terms of the threshold θ, the constellation symbol A and the $Q(x)$ function, where the latter gives the probability that a standard Gaussian random variable takes values greater than x. Note that an error will occur when $X_k = A$ and $Y_k < \theta$ or when $X_k = -A$ and $Y_k > \theta$. [4]

END OF PAPER

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