Version NS/4

EGT1
ENGINEERING TRIPOS PART IB

Friday 7 June 2019 2 to 4.10

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

*Answer not more than **two** questions from this section*

1 Consider the force field, $\mathbf{F} = xz^2\mathbf{i} + \left(x^2y - z^3\right)\mathbf{j} + \left(2xy + y^2z\right)\mathbf{k}$, in the Cartesian coordinate system (x, y, z) with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

(a) Evaluate $\nabla \cdot \mathbf{F}$. What are the shapes of iso-surfaces of $\nabla \cdot \mathbf{F} = C$ for different values of the constant C? [4]

(b) Evaluate $\nabla \times \mathbf{F}$. [4]

(c) Consider a hemispherical volume of radius R centred at $(0, 0, 0)$ with base surface S_1 and convex surface S_2. Find the total flux of $\mathbf{F}$ through the surfaces bounding this volume. [7]

(d) Find the work done by a particle experiencing the force field $\mathbf{F}$ as it moves along the circle bounding the base surface S_1 of the hemisphere. [7]

(e) Evaluate the fluxes of $\mathbf{F}$ through the surface S_1 and the convex surface S_2. [3]

2 (a) Consider the velocity field $\mathbf{U} = (2xy+3)\,\mathbf{i} + \left(x^2 - 4z\right)\mathbf{j} - 4y\mathbf{k}$, in the Cartesian coordinate system (x, y, z) with unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$.

(i) Show that the velocity field is conservative. [4]

(ii) Find the scalar potential field for $\mathbf{U}$. [4]

(iii) Evaluate the integral $\int \mathbf{U} \cdot d\mathbf{r}$ from $(3, -1, 2)$ to $(2, 1, -1)$ along paths of your choice. [3]

(b) Consider a horizontal circular plate of radius R and thickness t placed at the origin $(0, 0, 0)$. The plate is made of a composite material with a density distribution of $\rho\,(x, y, z) = \sqrt{x^2 + y^2}$ and has a hole of radius $R1$ at the centre.

(i) What is the total mass of the plate? [4]

(ii) Evaluate the integral $\int \left(x^2 + y^2\right)^{3/2}\, dv$ for the plate. What is the physical meaning of this integral? [6]

(iii) The radius of gyration is the distance from the axis of rotation at which, if the whole mass of the body were to be concentrated would result in the same moment of inertia as for the distributed mass. The axis of rotation is along $\mathbf{k}$. Calculate the radius of gyration for the plate. [4]

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3 The displacement, $y(x,t)$, of a vibrating string fixed at $x = 0$ and $x = L$ satisfies the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

At time $t = 0$, the string is displaced but held stationary so that $\partial y/\partial t = 0$.

(a) Using the method of separation of variables, deduce that

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\omega_n t \ \sin\left(\frac{n\pi x}{L}\right)$$

gives the displacement of the vibrating string. [10]

(b) What is the relationship between ω_n and c? [2]

(c) Discuss how you would determine the coefficients A_n. [3]

(d) Show that

$$A_1 = \frac{8h}{\pi^2} \qquad \text{and} \qquad A_3 = \frac{-8h}{9\pi^2}$$

if the string is displaced initially as shown in Fig. 1. [10]

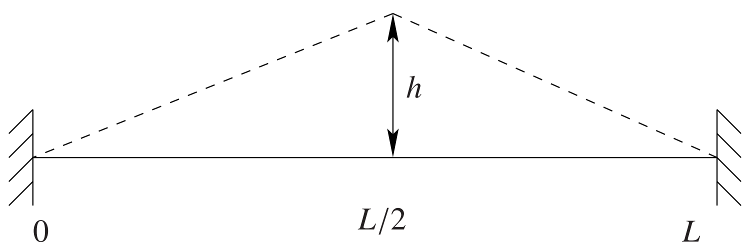

Fig. 1

SECTION B

*Answer not more than **two** questions from this section*

4 (a) Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & k^2+1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Solve $\mathbf{Ax} = \mathbf{b}$ for all values of k for which solutions exist. [5]

(b) Factor $\mathbf{A}$ into a lower triangular $\mathbf{L}$ and an upper triangular $\mathbf{U}$, and solve $\mathbf{Ax} = \mathbf{b}$ for the three different $\mathbf{b}$ given below:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \mathbf{b_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{b_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{b_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and then verify that your solutions $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are the three columns of $\mathbf{A}^{-1}$. [6]

(c) Can every 2×2 matrix $\mathbf{A}$ be factored into a lower triangular $\mathbf{L}$ times an upper triangular $\mathbf{U}$ matrices with elements on the diagonals not all zero?
For

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

find $\mathbf{L}$ and $\mathbf{U}$ (where possible). [8]

(d) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 6 \end{pmatrix}$$

Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal. [6]

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5 (a) Let

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 4 \end{pmatrix} \quad \mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$$

For $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ determine if they are eigenvectors of $\mathbf{A}$ and if so, find the corresponding eigenvalues. [4]

(b) Let

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$$

Calculate $\mathbf{A}^{100}$. [4]

(c) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

Find the best fit solution $\widehat{\mathbf{v}}$ for $\mathbf{Av} = \mathbf{b}$, the projection $\mathbf{p}$ of $\mathbf{b}$ onto the column space of $\mathbf{A}$, and the projection matrix $\mathbf{P}$ such that $\mathbf{p} = \mathbf{Pb}$ for any $\mathbf{b}$. [7]

(d) Let $\mathbf{R}$ be a 3×3 orthogonal matrix such that $det(\mathbf{R}) = \mathbf{1}$. Prove that 1 is an eigenvalue of $\mathbf{R}$ without using the characteristic equation. [10]

6 Let X be the delay in minutes of the train Oxford-London departing from Oxford at 7:45, and scheduled to arrive in London at 8:30. Assume that X is a random variable with probability density function

$$f(x) = \begin{cases} kx\exp\left(-\dfrac{x^2}{2}\right) & \text{for } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where k is an unknown constant.

(a) Find the correct value of k. [5]

(b) Find the probability that the train arrives in London after 8:33 but before 8:35. [7]

(c) Find the mean arrival time of the train. [5]

(d) Mary is a commuter and uses the 7:45 Oxford-London train 5 days per week. She will be late at work if the delay of the train is more than 5 minutes. Furthermore she will be blamed if she arrives late more than 2 times in 2 weeks. Assuming that the delay of the trains on different days is an independent random variable, find the probability that Mary will be not blamed in the next 2 weeks. [8]

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Numerical Answers - 2019

4. (a) $k \neq \pm\sqrt{2}$ $\quad x = \frac{1}{k^2-2}\begin{pmatrix} k^2+4 \\ k^2 \\ -2 \end{pmatrix}$

(b) $x_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(d) $\mathbf{P} = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

5. (a) x_1 is eigenvector of *A* with eigenvalue of 4, x_2 is not eigenvector of *A*, x_3 is eigenvector of *A* with eigenvalue of 3

(b) $2^{100}I$

(c) $\hat{\mathbf{v}} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, $\mathbf{p} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$

6. (a) $k = 1$, (b) 1.11%, (c) 8:31:15

(d) $P(Y \leq 2) = (1-e^{-12.5})^{10} + 10e^{-12.5}(1-e^{-12.5})^9 + 45e^{-25}(1-e^{-12.5})^8$