EGT1
ENGINEERING TRIPOS PART IB

Friday 11 June 2021 13.30 to 15.40

Paper 7

MATHEMATICAL METHODS

*This is an **open book** exam.*

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the top sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Your have access to the Engineering Data Book, online or as your hard copy

10 minutes reading time is allowed for this paper at the start of the exam.

The time allowed for scanning/uploading answers is 20 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

SECTION A

Answer not more than ***two*** *questions from this section*

1 (a) Fig. 1 shows a cone with a base surface S_1 and conical surface S_2. The base is defined as $x^2 + y^2 \leq a^2$, $z = 0$ and is bounded by the curve C. The cone is of height h.

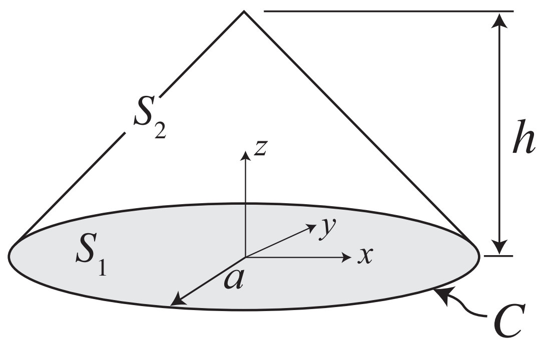

Fig. 1

A vector field $\mathbf{F}$ is given by:

$$\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} + 5\,\mathbf{k}$$

where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the unit vectors associated with the Cartesian coordinates (x, y, z).

(i) Confirm that $\mathbf{F}$ is solenoidal, calculate $\nabla\times\mathbf{F}$, and find the flux of $\nabla\times\mathbf{F}$ through S_1. [3]

(ii) What is the flux of $\mathbf{F}$ out through surfaces S_1 and S_2? [3]

(iii) Convert $\mathbf{F}$ into cylindrical polar coordinates. Hence sketch the field lines of $\mathbf{F}$ and evaluate $\oint \mathbf{F}\cdot d\mathbf{r}$ along curve C. [6]

(iv) Determine the flux of $\nabla\times\mathbf{F}$ through S_2 using Stokes's theorem and also by using Gauss's theorem. [4]

(b) A vector field $\mathbf{G}$ is defined, in Cartesian coordinates, by,

$$\mathbf{G} = \mathbf{i} + \sin x\,\mathbf{j} + \cos x\,\mathbf{k}.$$

Find the equations that describe the field lines of $\mathbf{G}$. Hence write down the equations for the field line that passes through $(0, 0, 0)$ and sketch this field line. [9]

2 (a) Consider the two-dimensional vector field $\mathbf{A} = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$.

(i) Calculate the line integrals from $(0,0)$ to $(1,1)$ along the paths

$$(0,0) \to (1,0) \to (1,1)$$

and

$$(0,0) \to (0,1) \to (1,1).$$

[3]

(ii) Show that a scalar potential must exist for $\mathbf{A}$ and find the potential. [4]

(iii) Find the equation for the field lines of $\mathbf{A}$. [3]

(b) Laplace's equation in two dimensions for temperature, T, is given below:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Solve this equation for the domain shown in Fig. 2, subjected to the boundary conditions given below:

$$\begin{aligned} T(0,y) &= 0, && \text{for} \quad 0 \le y \le b \\ T(x,0) &= 0, && \text{for} \quad 0 \le x \le a \\ T(a,y) &= 0, && \text{for} \quad 0 \le y \le b \\ T(x,b) &= T_0 \sin(\tfrac{\pi x}{a}), && \text{for} \quad 0 \le x \le a \end{aligned}$$

Note, T_0 is a constant and a and b are the width and height of the domain, respectively. [15]

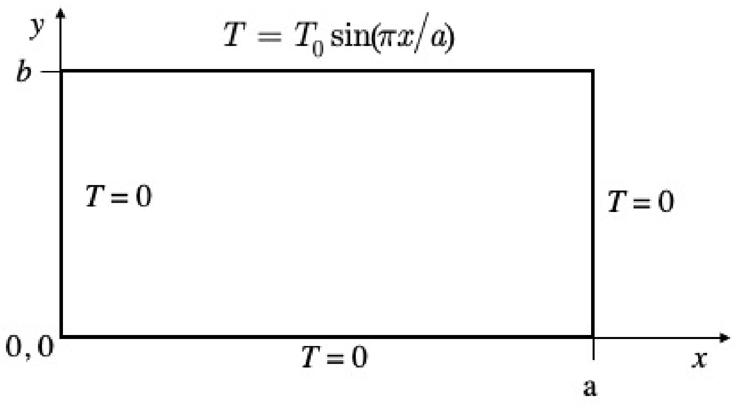

Fig. 2

3 The concentration (per unit mass) of dye in a fluid is ϕ. The dye is transported via diffusion and convection. The diffusive flux is given by $-\gamma\nabla\phi$, where γ is a constant, and the convective flux by $\rho\phi\mathbf{u}$, where ρ is the fluid's density and $\mathbf{u}$ its velocity.

(a) Using physical reasoning show that

$$\frac{\partial}{\partial t}\int_V \rho\phi dV = -\int_S \rho\phi\mathbf{u}\cdot d\mathbf{S} - \int_S (-\gamma\nabla\phi)\cdot d\mathbf{S}$$

where S is the surface bounding the volume V. Also $d\mathbf{S}$ is a vector representing an element of the surface, with a direction defined as positive pointing out of the volume. The integral on the left represents the rate of accumulation (or loss) of ϕ in the volume V. [5]

(b) Use Gauss's theorem to express the surface integrals in (a) as volume integrals. [3]

(c) If density is constant, show that

$$\rho\frac{\partial\phi}{\partial t} + \rho\mathbf{u}\cdot\nabla\phi = \gamma\nabla^{\mathbf{2}}\phi.$$

[7]

(d) If the system is one-dimensional and the fluid is stationary, a possible solution is $\phi(x,t)$ is $\phi(\eta)$, where,

$$\eta = \frac{x}{\sqrt{2\gamma t/\rho}}.$$

Find the ordinary differential equation that governs ϕ. [10]

SECTION B

*Answer not more than **two** questions from this section*

4 (a) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Find the eigenvalues of $\mathbf{A}$, an invertible matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is diagonal, and compute $\mathbf{B}$. Make sure to write down the intermediate steps which are performed in order to produce the final answer. [7]

(b) Show how the parameters C, D, E of the parabola $y = C + Dx + Ex^2$ that comes closest (least squares error) to the values below can be found:

$$\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \quad \mathbf{x} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

[7]

(c) If λ and μ are the eigenvalues of a 2×2 matrix $\mathbf{A}$, find the eigenvalues of matrix $\mathbf{A} - 2\mathbf{I}$. [4]

(d) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Find an orthogonal matrix $\mathbf{Q}$ (*i.e.*, $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$) and an upper triangular matrix $\mathbf{R}$ such that

$$\mathbf{A} = \mathbf{QR}$$

Make sure to write down the intermediate steps which are performed in order to produce the final answer. [7]

5 (a) Let

$$\mathbf{A} = \begin{pmatrix} k & -1 & k-1 \\ 1 & 1 & 1-k \\ 2 & 1 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} k \\ k+2 \\ 1 \end{pmatrix}$$

Solve $\mathbf{Ax} = \mathbf{b}$ for all values of k for which solutions exist. [10]

(b) Let

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{pmatrix}$$

Factor $\mathbf{A}$ into $\mathbf{LU}$ and solve $\mathbf{Ax} = \mathbf{b}$ for the three right-hand sides:

$$\mathbf{b_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{b_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{b_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Verify that your solutions $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are the three columns of A^{-1}. Make sure to write down the intermediate steps which are performed in order to produce the final answer. [8]

(c) Let

$$\mathbf{A}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

find a vector $\mathbf{v}$ and a scalar λ, that serve as an eigenvector and eigenvalue respectively for $\mathbf{A}(t)$, that are independent of t. [7]

6 (a) Let X be a discrete random variable with the following probability mass function:

$$P_X(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(i) Compute the expectation of X.

(ii) Compute the variance of X.

(iii) If $Y = (X - d)^2$, find the expectation of Y for $d = 2$.

(iv) Find the value of d for which the variance of Y is smallest. You may use the result that $Var(A + B) = Var(A) + Var(B) + 2Cov(A, B)$, where A and B are real-valued random variables.

[9]

(b) Let V be a continuous uniform random variable taking values in $(0, 2)$ and define $X = \ln V$.

(i) Compute $\mathbb{P}(X \leq \ln 2)$.

(ii) Compute the cumulative probability function $F_X(x) = \mathbb{P}(X \leq x)$.

(iii) Compute the expectation and the variance of X.

(iv) Compute and identify the probability density function of $Y = \ln 2 - X$. [16]

END OF PAPER

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