EGT1
ENGINEERING TRIPOS PART IB

Tuesday 13 June 2023   2 to 4.10

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer not more than **two** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

*Answer not more than **two** questions from this section*

1 Consider the region $\mathfrak{R}$ bounded by the curves $xy = 1$, $xy = 2$, $xy^3 = 3$, and $xy^3 = 9$.

(a) Sketch $\mathfrak{R}$. [5]

(b) Evaluate the area of $\mathfrak{R}$. [10]

(c) If $d\mathbf{l}$ is the infinitesimal line element along the boundary of $\mathfrak{R}$, find $\oint \mathbf{F} \cdot d\mathbf{l}$ if

$$\mathbf{F} = \frac{2}{5}(xy^5)^{1/2}\,\mathbf{i} + (xy)^{3/2}\,\mathbf{j}$$

[10]

2 (a) State the divergence theorem and discuss its range of validity. [5]

(b) Prove the following expression, which is known as Green's first identity:

$$\int_V (\phi \nabla^2 \psi + \nabla\phi \cdot \nabla\psi)\, dV = \oiint_S (\phi \nabla \psi) \cdot \mathbf{n}\, dA$$

where the integration is over the volume V enclosed by any closed surface S, $\mathbf{n}$ denotes the unit vector normal to the elemental area dA, and ϕ and ψ are scalar functions. [5]

(c) Consider a cylinder with radius R, whose axis is the $z-$axis of a Cartesian coordinate system, and whose circular faces are the planes $z = 0$ and $z = H$. Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(i) Using the divergence theorem, and not otherwise, evaluate $\oiint \mathbf{F} \cdot \mathbf{n}\, dA$ over the closed surface of the cylinder. [5]

(ii) Evaluate again $\oiint \mathbf{F} \cdot \mathbf{n}\, dA$ by direct integration considering separately all surfaces of the cylinder, and hence show that the divergence theorem is satisfied. [5]

(iii) A cylindrical volume whose axis is also the $z-$axis, and with a height H and radius $R/2$, is now removed from the original cylinder. Evaluate the flux of $\mathbf{F}$ over all enclosing surfaces of the remaining hollow cylinder. [5]

3 In a two-dimensional heat transfer problem, the steady-state equation for the normalised temperature T is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

where $f(x, y)$ is a localised source of heat. We seek solutions for $T(x, y)$ in the domain $0 \leq x \leq 1$, $0 \leq y \leq 1$. The boundary conditions are $T(0, y) = T_0$, $T(1, y) = T_0$, $T(x, 1) = T_0$, and $T(x, 0) = T_0\big(1 + \sin(2\pi x)\big)$.

(a) Use separation of variables to find an analytical expression for $T(x, y)$ for the case $f(x, y) = 0$. [15]

(b) The heat flow is defined as $\mathbf{q} = -\nabla T$. If $f(x, y) = \sin(4\pi x)\sin(4\pi y)$, find the total heat flow $\oint \mathbf{q} \cdot d\mathbf{n}$, where the integral is taken along the boundary of the domain and $\mathbf{n}$ is the unit vector normal to the domain's boundary. [10]

SECTION B

*Answer not more than **two** questions from this section*

4 (a) Let

$$\mathbf{A} = \begin{pmatrix} 3 & 4 & 0 & 1 \\ 6 & 4 & 2 & t \\ 9 & 0 & 6 & 1 \end{pmatrix}$$

Factor **A** into **LU**, and find the value of t for which **A** is of rank 2. [6]

(b) Let

$$\mathbf{A} = \begin{pmatrix} 1 & k+1 & 1 \\ k-3 & 1 & 1 \\ 1 & 1 & 1-k \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} k+5 \\ 1 \\ 5 \end{pmatrix}$$

Solve $\mathbf{Ax} = \mathbf{b}$ for all values of k for which solutions exist. [7]

(c) Let $\mathbf{C} - 2\mathbf{I} = -\mathbf{C}^2$ where **C** is an $n \times n$ matrix. Identify all possible eigenvalues that **C** can have. [4]

(d) Let

$$\mathbf{A} = \begin{pmatrix} a & b & b & \cdots & b \\ c & a & 0 & \cdots & 0 \\ c & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & 0 & 0 & \cdots & a \end{pmatrix}$$

be an $n \times n$ matrix and $b > 0$, $c > 0$. Find the eigenvalues, set of eigenvectors, and determinant of **A**. [8]

5 (a) Let

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

(i) Find a QR decomposition of **A**, making clear your intermediate calculation steps. [4]

(ii) Using the obtained QR decomposition, find the least squares solution for $\mathbf{Ax} = \mathbf{b}$. [3]

(b) Let

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 1 & -a \end{pmatrix}$$

Calculate $\mathbf{A}^{20}$, making clear your intermediate calculation steps. [5]

(c) Find an orthonormal basis for the plane $2x - 3y + z = 0$ in $\mathbb{R}^3$. [6]

(d) Let

$$\mathbf{A} = \begin{pmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(i) Find a singular value decomposition of **A**, for all real values of a, making clear your intermediate calculation steps. [5]

(ii) Identify the basis for each of the four fundamental subspaces of **A**. [2]

6 (a) Teams A and B are playing a game which consists of maximum 7 rounds. A team wins the game if it wins 4 rounds, after which no further rounds, if any left, are played. Rounds cannot end in a tie.

(i) The probability that A wins any round is p. What is the probability that the teams will play all seven rounds? [3]

(ii) The probability that A wins is now different when it plays in the home field (p) and in the visiting field (r). The teams alternate between home and away rounds. What is the probability that the teams will play all seven rounds? [4]

(b) A transmitter sends out either a 1 with probability p, or a 0 with probability $1 - p$, independent of earlier transmissions. If the number of transmissions within a given time interval has a Poisson probability mass function (PMF) with parameter λ, show that the number of 1s transmitted in that same time interval has a Poisson PMF with parameter $p\lambda$. [6]

(c) Let X and Y be two continuous random variables with probability density function (PDF) f_X and f_Y, respectively. The random variable Z is equal to X with probability p, and to Y with probability $1 - p$.

(i) Show that the PDF of Z is $f_Z = pf_X + (1 - p)f_Y$. [2]

(ii) Let $f_X(x) = \lambda e^{\lambda x}$ if $x < 0$ ($f_X(x) = 0$ otherwise), and $f_Y(y) = \lambda e^{-\lambda y}$ if $y \geq 0$ ($f_Y(y) = 0$ otherwise), with $\lambda > 0$. Compute the cumulative distribution function F_Z of the random variable Z. [2]

(iii) Compute the mean and variance of the random variable Z. [3]

(d) Let $X_1, X_2, \ldots, X_n$ be n mutually independent random variables, all identically, exponentially distributed as $\mathrm{Exp}(\lambda)$, with $\lambda > 0$. Let $S = \max(X_1, X_2, \ldots, X_n)$. After justifying that $\mathbb{P}[S \leq s] = \mathbb{P}[X_1 \leq s \cap X_2 \leq s \cap \cdots \cap X_n \leq s]$, compute the probability density function of S. [5]

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1.b) $\frac{1}{2}\ln 3$

1.c) $-\frac{1}{2}(3-\sqrt{3})$

2.c.i) $3\pi R^2 H$

2.c.iii) $\frac{13}{4}\pi R^2 H$

3.a) $T(x,y) = T_0\big(1+\sin(2\pi x)\cosh(2\pi)\sinh[2\pi(1-y)]\big)$

3.b) $\oint \mathbf{q}\cdot d\mathbf{n} = 0$

4.a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 0 & 1 \\ 0 & -4 & 2 & t-2 \\ 0 & 0 & 0 & 4-3t \end{pmatrix}$; $t = 2$

4.b) $\mathbf{x} = \frac{1}{k}\begin{pmatrix} 0 \\ k+4 \\ -4 \end{pmatrix}$ if $k \neq 0, 3$; $\mathbf{x} = \begin{pmatrix} 1 \\ 4-z \\ z \end{pmatrix}$ if $k = 0$; $\mathbf{x} = \begin{pmatrix} 4+3z \\ 1-z \\ z \end{pmatrix}$ if $k = 3$

4.c) $\lambda = 1, -2$

4.d) $\lambda = a, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ -1 \end{pmatrix}$; $\lambda = a \pm \sqrt{(n-2)bc}, \begin{pmatrix} \pm\sqrt{(n-2)b/c} \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

5.a.i) $\mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{15}} \\ 0 & \frac{\sqrt{2}}{\sqrt{5}} & -\frac{2}{\sqrt{15}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{15}} \\ 0 & \frac{\sqrt{2}}{\sqrt{5}} & \frac{3}{\sqrt{15}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{5}}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{5}} \\ 0 & 0 & \frac{3}{\sqrt{15}} \end{pmatrix}$

5.a.ii) $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

5.b) $\mathbf{A}^{20} = a^{20}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5.c) $\frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{182}}\begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix}$

5.d.i) $\mathbf{A} = \begin{pmatrix} 0 & \frac{a}{|a|} & \frac{1}{\sqrt{1+a^2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{|a|} & -\frac{a}{\sqrt{1+a^2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & |a| & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ if $0 < |a| < 1$;

$$\mathbf{A} = \begin{pmatrix} \frac{a}{|a|} & 0 & \frac{1}{\sqrt{1+a^2}} \\ 0 & 1 & 0 \\ \frac{1}{|a|} & 0 & -\frac{a}{\sqrt{1+a^2}} \end{pmatrix} \begin{pmatrix} |a| & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } 1 < |a|;$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } a = 0$$

6.a.i) $20p^3(1-p)^3$

6.a.ii) $p^3(1-r)^3 + 9p^2(1-p)r(1-r)^2 + 9p(1-p)^2r^2(1-r) + (1-p)^3r^3$

6.c.ii) $\begin{cases} pe^{\lambda z} & z < 0 \\ 1-(1-p)e^{-\lambda z} & z \geq 0 \end{cases}$

6.c.iii) $\mathbb{E}[\mathrm{Z}] = \frac{1-2p}{\lambda}$; $\mathrm{Var}[\mathrm{Z}] = \frac{2-(1-2p)^2}{\lambda^2}$

6.d) $n\lambda e^{-\lambda s}(1-e^{-\lambda s})^{n-1}$