EGT2
ENGINEERING TRIPOS PART IIA

Tuesday 23 April 2019   9.30 to 12.40

Module 3A1

FLUID MECHANICS I

*Answer not more than **five** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Attachments:

- Incompressible Flow Data Card (2 pages);
- Boundary Layer Theory Data Card (1 page);
- 3A1 Data Sheet for Applications to External Flows (2 pages).

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) In a radial polar (r, θ) co-ordinate system, a line vortex with circulation Γ is placed at $(R, 0)$ in an otherwise stationary 2D flow. The fluid is inviscid, irrotational, and incompressible. Show that, on a circle of radius R centred on the origin, the vortex induces a velocity:

$$u_r = -\frac{\Gamma}{4\pi R}\cot\left(\frac{\theta}{2}\right) \; ; \quad u_\theta = \frac{\Gamma}{4\pi R} \; .$$

[30%]

(b) A 2D circular vortex sheet with radius R and constant circulation per unit length γ, shown in Fig. 1, has total circulation Γ. Calculate u_r and u_θ of the sheet itself. Describe its motion. [20%]

(c) By considering a closed contour just outside the sheet, or otherwise, calculate u_r and u_θ just outside the vortex sheet. [10%]

(d) Using Stokes' theorem, or otherwise, calculate the velocity jump across the vortex sheet. [10%]

(e) Sketch $u_\theta(r)$ for this flow and for a line vortex with strength Γ centred on the origin. [20%]

(f) Discuss the advantages and disadvantages, if any, of using this vortex sheet to model the flow around a rotating cylinder. [10%]

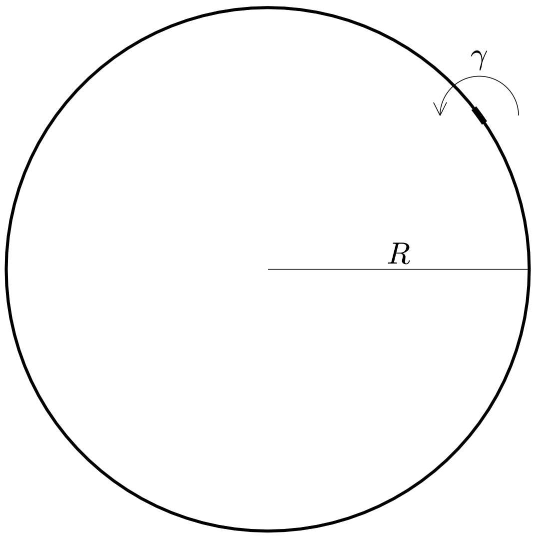

Fig. 1

2 A line vortex of strength Γ is placed at $(x, y) = (0, a)$ above a flat surface at $y = 0$ in a flow that, at infinity, has speed U in the x-direction. The flow is inviscid, irrotational, and incompressible.

(a) Find U for which the vortex remains stationary and write down the complex potential, $F(z)$, for this flow. [10%]

(b) Show that the stagnation points lie at $z = \pm\sqrt{3}a$ and sketch the streamlines of this flow. [20%]

(c) Perform a conformal mapping such that this system models the flow around a vortex trapped in a corner, as shown in Fig. 2. Write down the complex potential in the new co-ordinate system. Find the positions of the stagnation points and sketch the streamlines of this flow. [20%]

(d) Find the positions of the points with lowest pressure on the surfaces. [40%]

(e) Sketch the image system for this flow. Does the vortex remain stationary? [10%]

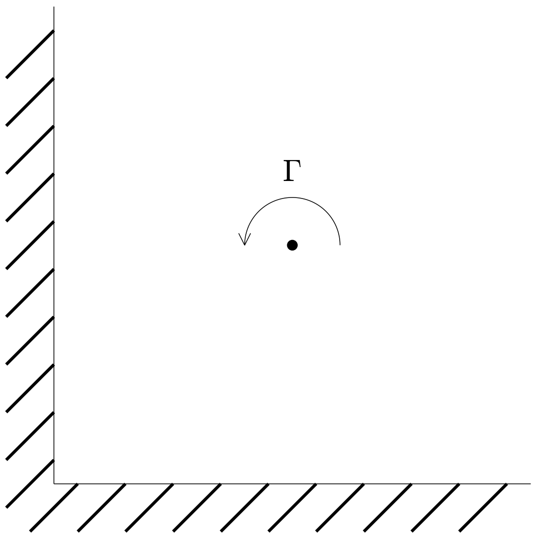

Fig. 2

3 The Navier–Stokes equation in a reference frame rotating at angular velocity $\mathbf{\Omega}$ is:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x}) = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} \qquad (1)$$

(a) Using the identity $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x}) = -\nabla((\mathbf{\Omega}\times\mathbf{x})^2/2)$, where $\mathbf{x}$ is the position vector, show that the fourth term in Eq. (1) can be absorbed into the pressure term by defining a 'reduced pressure', P. Interpret this result physically. [10%]

(b) For velocities of order U over lengthscales of order L, estimate the order of the ratio of the inertial term $(\mathbf{u}\cdot\nabla\mathbf{u})$ to the Coriolis term $(2\mathbf{\Omega}\times\mathbf{u})$. [10%]

(c) A viscous fluid moves steadily between a flat plate at $z = 0$, rotating at Ω, and a parallel flat plate at $z = Z$, rotating slightly faster. The flow has Re $\gg 1$ and consists of an inviscid interior sandwiched between two boundary layers adjacent to the plates. In Cartesian co-ordinates (x, y, z) with $\mathbf{u} = (u, v, w)$ the equations governing the velocity $\mathbf{u}$ in the rotating frame are:

$$2\mathbf{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{u}$$
$$\nabla\cdot\mathbf{u} = 0$$

Write down expressions for the three components of ∇P in the inviscid interior in terms of the inviscid flow $\mathbf{u}_I = (u_I, v_I, w_I)$. Deduce how $\mathbf{u}_I$ varies with z in this flow. Comment on the consequences of this result. [30%]

(d) Assuming that variations of $\mathbf{u}$ within the boundary layers are much more rapid in z than in x or y, show that

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$
$$2\Omega u = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}$$

[10%]

(e) Determine $\partial P/\partial x$ and $\partial P/\partial y$ within the boundary layer from their values in the inviscid interior. Then, with $\mathrm{i} = \sqrt{-1}$, define $f = u - u_I + \mathrm{i}(v - v_I)$ and derive an equation for $\mathrm{d}^2 f/\mathrm{d}z^2$. Find the general solution to f and the order of the boundary layer thickness in terms of ν and Ω. [40%]

4 Consider a uniform flow with x-velocity U passing a 2D streamlined body at high Reynolds number. Downstream of the body there is a thick wake in which $\mathbf{u}$ varies much more rapidly with y than with downstream distance x.

(a) Stating your assumptions, apply the arguments of boundary layer theory to deduce an approximate equation for the velocity (u, v) in the wake and state the boundary conditions for u. [10%]

(b) The flow sufficiently far downstream from the body can be approximated by $u = U + u_1$, where $|u_1| \ll U$. Show that the momentum equation approximates to

$$U\frac{\partial u_1}{\partial x} = \nu\frac{\partial^2 u_1}{\partial y^2} .$$

and confirm that $\int_{-\infty}^{\infty} u_1 \, \mathrm{d}y$ does not vary in x. [10%]

(c) Seek a similarity solution of the form $u_1 = F(x)f(\eta)$; $\eta = y/g(x)$ and show that $F(x) \propto 1/g(x)$. [20%]

(d) Show that, in order to have a similarity solution, Ugg'/ν must be constant, where $'$ denotes a derivative. Take this constant to be 1. [20%]

(e) Deduce the similarity equation $f'' + f + \eta f' = 0$. [20%]

(f) Show that $u_1 = \frac{A}{x^{1/2}}\mathrm{e}^{-Uy^2/4\nu x}$, where A is a constant. [20%]

5 (a) Thwaites defined two dimensionless parameters l and m for laminar boundary layers, such that

$$l = \frac{\theta}{U}\left(\frac{\partial u}{\partial y}\right)_{\text{wall}} = \frac{\theta}{U}\frac{\tau_w}{\mu}\,, \qquad m = \frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_{\text{wall}}$$

where $U = U(x)$ is the free-stream velocity, μ is the fluid's dynamic viscosity, τ_w is the wall shear stress, and θ is the momentum thickness of the boundary layer.

(i) Show that $m = -\frac{\theta^2}{\nu}\frac{dU}{dx}$, where ν is the fluid's kinematic viscosity. [10%]

(ii) Use the momentum integral equation to deduce that

$$U\frac{d(\theta^2)}{dx} = 2\nu[(H+2)m + l]\,,$$

where H is the shape factor. [20%]

(iii) Assuming that $2[(H+2)m+l] = 0.45 + 6m$, deduce that

$$\theta^2(x) = \theta^2(0)\left(\frac{U(0)}{U(x_1)}\right)^6 + \frac{0.45\nu}{U^6(x)}\int_0^x U^5(x')\mathrm{d}x'\,,$$

where x is the location of interest. [20%]

(b) Consider the inviscid flow around a cylinder of radius a described by the velocity potential $\phi(r,\alpha) = -U_\infty(r + a^2/r)\sin\alpha$, where α is the angle to the x-axis in radial polar co-ordinates.

(i) Show that the velocity on the surface of the cylinder is $u_\alpha = -2U_\infty(x/a)$. [20%]

(ii) For $|x| \ll a$, find a leading order approximation for the square of the momentum thickness, $\theta^2\,(x)$, in the region of the front stagnation point $(0,a)$. [15%]

(iii) Comment briefly on any notable features of your expression. [15%]

6 A flexible aerofoil consists of two straight sections connected by a hinge at $x = c/2$, as shown in Fig.3.

(a) By expressing $-2\mathrm{d}y_c/\mathrm{d}x$ as a Fourier series in the usual manner for thin aerofoil theory, find the first three coefficients, g_0 , g_1, and g_2. You may assume that ψ is small. [40%]

(b) Calculate the sectional lift coefficient as a function of ψ and angle of attack α. You may assume that α is small. [10%]

(c) For the case of $\alpha = 0$, and for non-zero positive ψ, calculate the vortex sheet strength as a function of x/c and sketch the upper and lower surface pressure coefficient distributions. [25%]

(d) In order to achieve a certain level of lift, would it be better to increase α or ψ? Explain your answer. [25%]

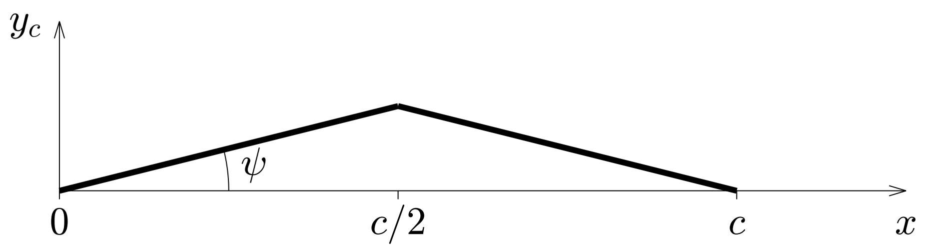

Fig. 3

7 An aircraft of wing-span $2s$ is entering a narrow 'urban canyon' of width $2b$ as sketched in Fig.4. The wing loading is elliptical and the aircraft is sufficiently far from the ground that the ground effect is negligible.

(a) Using the horseshoe vortex model, sketch the distribution of vortices modelling this scenario. [20%]

(b) If the aircraft speed does not change, does it need to change angle of attack when entering the canyon? [10%]

(c) Estimate the relative change in induced drag coefficient if $b = 2s$. [50%]

(d) What would your answer to (c) be if $b \to s$? Comment on the suitability of the horseshoe vortex model in this scenario. [20%]

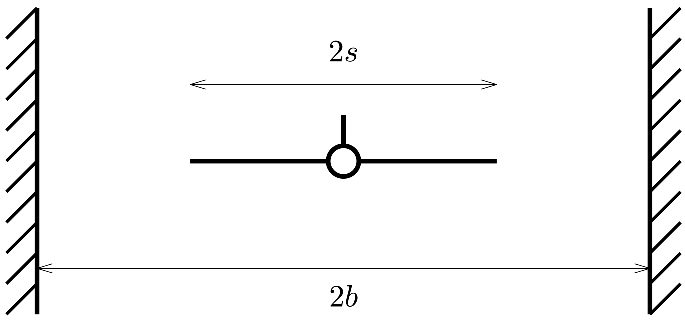

Fig. 4

8 (a) The Tucker 48 car (Fig. 5) designed in 1944 had its aerodynamic form designed in just 6 days without using a wind tunnel or computer. It had a suggested drag coefficient C_D = 0.27, which is better than many modern cars. Discuss how this low C_D value is obtained? Note that the engine is at the rear. [20%]

(b) Using approximate sketches show streamlines of the flow over the Tucker 48 and hence estimate the corresponding surface pressure distributions. Label the rotational and irrotational flow zones and areas likely to exhibit strong large-scale unsteadiness. [10%]

(c) Describe where cavity flows can be found on ground vehicles. Describe the aerodynamics of cavity flows and how this links to noise. [20%]

(d) Discuss how small-scale features can be optimised to reduce drag and noise of road vehicles. [20%]

(e) Quantitatively estimate the aerodynamic impact of wing mirrors on the engine power requirements of the Tucker 48 car. Assume that the wing mirrors increase the cross-sectional area of the car by 1% in the plane normal to its velocity. State all your assumptions and note the design implications. [20%]

(f) For a road vehicle, briefly discuss the trade-off between aerodynamic drag, noise, and cooling in relation to stagnation point location. [10 %]

Fig. 5

END OF PAPER

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Module 3A1: Fluid Mechanics I

INCOMPRESSIBLE FLOW DATA CARD

Continuity equation $\nabla \cdot \boldsymbol{u} = 0$

Momentum equation (inviscid) $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

Vorticity $\boldsymbol{\omega} = \text{curl } \boldsymbol{u}$

Vorticity equation (inviscid) $\dfrac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$

Kelvin's circulation theorem (inviscid) $\dfrac{D\Gamma}{Dt} = 0, \quad \Gamma = \oint \boldsymbol{u} \cdot \boldsymbol{dl} = \int \boldsymbol{\omega} \cdot \boldsymbol{dS}$

For an irrotational flow

velocity potential ϕ $\boldsymbol{u} = \nabla \phi$ and $\nabla^2 \phi = 0$

Bernoulli's equation for inviscid flow: $\dfrac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} = \text{constant}$ throughout flow field

TWO-DIMENSIONAL FLOW

Streamfunction ψ

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

Lift force Lift / unit length $= \rho U(-\Gamma)$

For an irrotational flow

complex potential *F(z)* $F(z) = \phi + i\psi$ is a function of $z = x + iy$

$$\frac{dF}{dz} = u - iv$$

TWO-DIMENSIONAL FLOW (continued)

Summary of simple 2 - D flow fields

	ϕ	ψ	$F(z)$	$\boldsymbol{u}$
Uniform flow (x - wise)	Ux	Uy	Uz	$u = U,\ v = 0$
Source at origin	$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$	$\frac{m}{2\pi}\ln z$	$u_r = \frac{m}{2\pi r},\ u_\theta = 0$
Doublet (x - wise) at origin	$-\frac{\mu\cos\theta}{2\pi r}$	$\frac{\mu\sin\theta}{2\pi r}$	$-\frac{\mu}{2\pi z}$	$u_r = \frac{\mu\cos\theta}{2\pi r^2},\ u_\theta = \frac{\mu\sin\theta}{2\pi r^2}$
Vortex at origin	$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$	$-\frac{i\Gamma}{2\pi}\ln z$	$u_r = 0,\ u_\theta = \frac{\Gamma}{2\pi r}$

THREE-DIMENSIONAL FLOW

Summary of simple 3 - D flow fields

	ϕ	$\boldsymbol{u}$
Source at origin	$-\frac{m}{4\pi r}$	$u_r = \frac{m}{4\pi r^2},\quad u_\theta = 0,\quad u_\psi = 0$
Doublet at origin (with θ the angle from the doublet axis)	$-\frac{\mu\cos\theta}{4\pi r^2}$	$u_r = \frac{\mu\cos\theta}{2\pi r^3},\quad u_\theta = \frac{\mu\sin\theta}{4\pi r^3},\quad u_\psi = 0$

Boundary Layer Theory Data Card

Displacement thickness;

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Momentum thickness;

$$\theta = \int_0^\infty \frac{(U-u)u}{U^2} dy = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U} dy$$

Energy thickness;

$$\delta_E = \int_0^\infty \frac{(U^2-u^2)u}{U^3} dy = \int_0^\infty \left(1 - \left(\frac{u}{U}\right)^2\right) \frac{u}{U} dy$$

$$H = \frac{\delta^*}{\theta}$$

Prandtl's boundary layer equations (laminar flow);

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

von Karman momentum integral equation;

$$\frac{d\theta}{dx} + \frac{H+2}{U}\theta\frac{dU}{dx} = \frac{\tau_w}{\rho U^2} = \frac{C_f'}{2}$$

Boundary layer equations for turbulent flow;

$$\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &= -\frac{1}{\rho}\frac{d\overline{p}}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \end{aligned}$$

3A1 Data Sheet for Applications to External Flows

Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure p_∞:

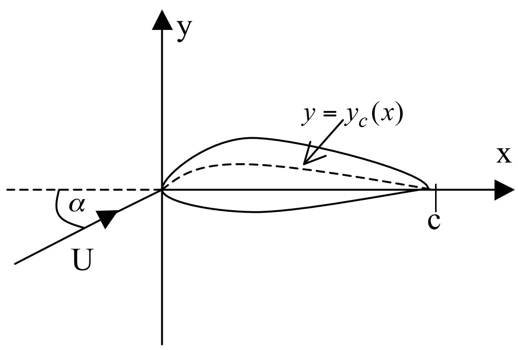
Pressure coefficient: $$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

Section lift and drag coefficients: $$c_l = \frac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c},\quad c_d = \frac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$$ (section chord c)

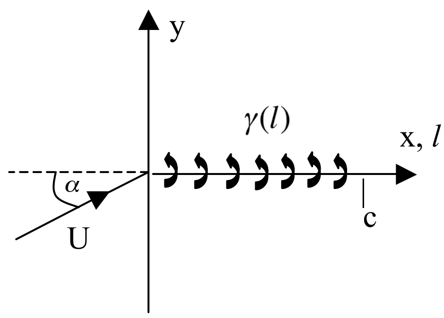
Wing lift and drag coefficients: $$C_L = \frac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S},\quad C_D = \frac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$$ (wing area S)

Thin Aerofoil Theory

Geometry



Approximate representation



Pressure coefficient: $c_p = \pm\gamma / U$

Pitching moment coefficient: $c_m = (\text{moment about } x = 0) / \frac{1}{2}\rho U^2 c^2$

Coordinate transformation: $l = c(1+\cos\phi)/2, \quad x = c(1+\cos\theta)/2$

Incidence solution: $$\gamma(l) = -2U\alpha \frac{1-\cos\phi}{\sin\phi},\ c_l = 2\pi\alpha,\ c_m = c_l/4$$

Camber solution: $$\gamma(l) = -U\left[g_0 \frac{1-\cos\phi}{\sin\phi} + \sum_{n=1}^{\infty} g_n \sin n\phi\right],\ \text{where}$$

$$g_0 = \frac{1}{\pi}\int_0^\pi \left(-2\frac{dy_c}{dx}\right) d\theta,\ g_n = \frac{2}{\pi}\int_0^\pi \left(-2\frac{dy_c}{dx}\right)\cos n\theta\, d\theta\,;$$

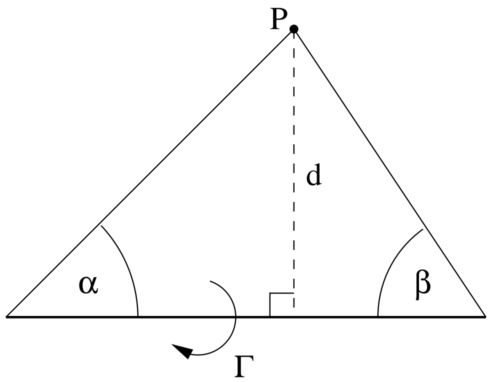
or, equivalently: $$-2\frac{dy_c}{dx} = g_0 + \sum_{n=1}^{\infty} g_n \cos n\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right),\ c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

Glauert Integral

$$\int_o^\pi \frac{\cos n\phi}{\cos\phi - \cos\theta} d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

Line Vortices



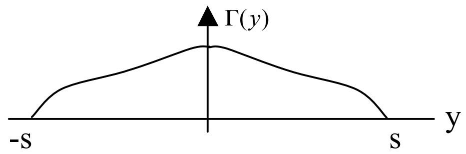
The Biot-Savart integral for a straight element of circulation Γ gives a contribution to the velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

Lifting-Line Theory

Spanwise circulation distribution:



Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y) dy$

Downwash angle: $\alpha_d(y) = \frac{1}{4\pi U} \int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y - \eta} \, d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y) \alpha_d(y) dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{\text{n odd}} G_n \sin n\theta$, with $y = -s\cos\theta$;

equivalently, $G_n = \frac{2}{\pi} \int_0^\pi \frac{\Gamma(y)}{Us} \sin n\theta \, d\theta$

Relation between lift and induced drag:

$$C_{Di} = (1+\delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \ldots$$

Elliptic lift distribution: $\Gamma(y) = \Gamma_0 \left(1 - \frac{y^2}{s^2}\right)^{1/2}$, $L = \frac{\pi}{2}\rho U \Gamma_0 s$, $\alpha_d = \frac{\Gamma_0}{4Us}$, $\delta = 0$