EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 24 April 2019 2 to 3.40

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C5 Dynamics and 3C6 Vibration data sheet (6 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 The motion of a rigid body is described by its linear momentum $\boldsymbol{p}$ and its moment of momentum $\boldsymbol{h}_\mathrm{P}$. The body is subject to an external force $\boldsymbol{F}^{(\mathrm{e})}$ and to an external couple $\boldsymbol{Q}^{(\mathrm{e})}$. Moments are taken about a general moving point P whose motion is described by the position vector $\boldsymbol{r}_\mathrm{P}$.

(a) Beginning with Newton's laws for a particle derive the standard results for the motion of a rigid body:

(i) $\boldsymbol{F}^{(\mathrm{e})} = \dot{\boldsymbol{p}}$ [25%]

(ii) $\boldsymbol{Q}^{(\mathrm{e})} = \dot{\boldsymbol{h}}_\mathrm{P} + \dot{\boldsymbol{r}}_\mathrm{P} \times \boldsymbol{p}$ [50%]

(b) Show that a special result holds when P coincides at all times with the centre of mass of the body. [10%]

(c) Show that a special result holds for moment of momentum about the contact point P when a ball is rolling on a horizontal rotating turntable. [15%]

2 A light rigid frame ABCD has four small particles each of mass m fixed at points A, B, C and D as shown in Fig. 1. Relative to Cartesian axes Oxyz the coordinates of the points are as follows: A is $(a,0,2a)$; B is $(a,0,0)$; C is $(-a,0,0)$ and D is $(-a,2a,0)$.

Find:

(a) the inertia matrix of the frame at O referred to axes (x,y,z); [30%]

(b) the (x,y,z) coordinates of the mass centre G of the frame; [10%]

(c) the inertia matrix at G referred to axes parallel to (x,y,z); [30%]

(d) the inertia matrix at B referred to axes parallel to (x,y,z); [20%]

(e) one of the principal moments of inertia at B. [10%]

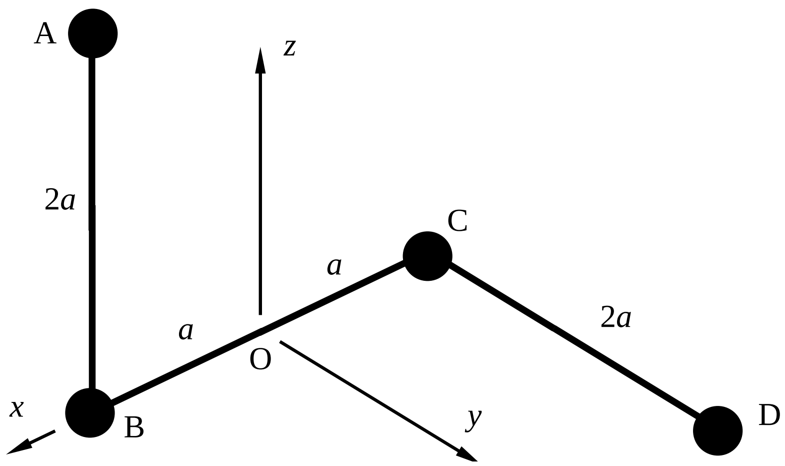

Fig. 1

3 (a) A solid ball of radius a amd mass m is rolling without slip on the inner surface of a vertical cylinder of radius R as shown in Fig. 2. The ball is in steady motion so that the contact point P moves in a horizontal plane at speed v_{P}.

(i) Use a no-slip condition at P to determine constraints on the angular velocity of the ball. [25%]

(ii) Find all components of the angular velocity of the ball. [25%]

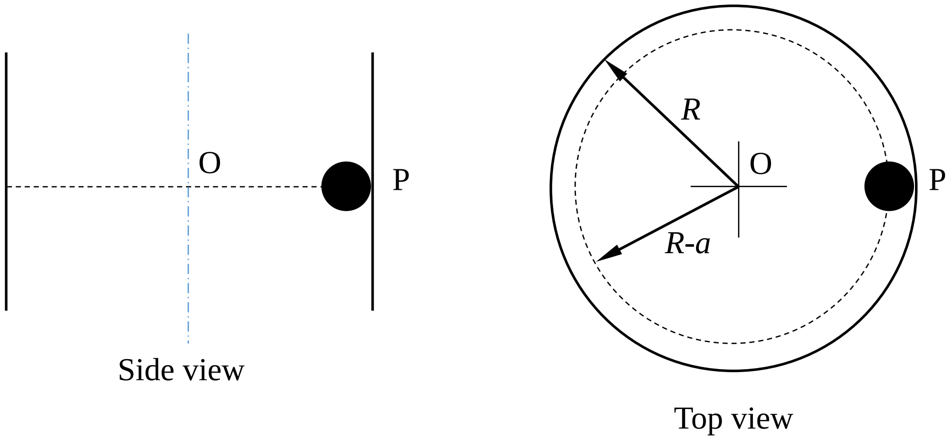

Fig. 2

(b) (i) Within the context of Lagrange's equation, explain why the generalised velocities $\dot{q}_i(t)$ and the generalised displacements $q_i(t)$ are considered to be independent variables, even though the velocities can clearly be obtained as the time derivatives of the displacements. [25%]

(ii) Give an example of a system in which the kinetic energy is a function of both the velocities and the displacements of the system. Show that in general the Lagrange equation for this type of system has the form

$$\sum_j \left\{ \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \ddot{q}_j + \frac{\partial^2 T}{\partial \dot{q}_i \partial q_j} \dot{q}_j \right\} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \quad , \qquad i = 1, 2, \ldots, N$$

where the symbols have their usual meaning. Will the second term on the left hand side of this equation provide damping to the system? [25%]

4 A chain of length L and mass m per unit length hangs freely under gravity from a fixed point. A horizontal excitation force $F(t)$ is applied to the base of the chain. Small lateral (horizontal) displacement $w(x,t)$ of the chain is modelled by a two term series so that

$$w(x,t) = (x/L)q_1(t) + (x/L)^2 q_2(t)$$

where x is measured downwards from the top of the chain and $q_1(t)$ and $q_2(t)$ are generalised coordinates. The potential energy of the chain is given by

$$V = \frac{1}{2}\int_0^L P(x)\left(\frac{\partial w}{\partial x}\right)^2 dx$$

where $P(x)$ is the tension in the chain in the *equilibrium position*. The acceleration due to gravity is g.

(a) Use Lagrange's equation to show that the two equations for small motion of the chain can be expressed in matrix forms as

$$mL\begin{bmatrix} 1/3 & 1/4 \\ 1/4 & 1/5 \end{bmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + mg\begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} F \\ F \end{pmatrix}$$ [40%]

(b) If a single degree of freedom model is employed, using $q_1(t)$ alone, demonstrate that the resulting equation of motion is identical to that of a hanging rigid rod, and can be expressed in terms of the moment of inertia and the mass of the rod. Derive an expression for the natural frequency of the chain using this model. [10%]

(c) Derive the natural frequencies of the chain using the two degrees of freedom $q_1(t)$ and $q_2(t)$. [35%]

(d) Exact results for the first two natural frequencies are known for the chain, and they are given by $\omega_1 = 1.202\sqrt{g/L}$ and $\omega_2 = 2.76\sqrt{g/L}$. Comment on the accuracy of your results in the light of these values, and explain how the model could be further improved. [15%]

END OF PAPER

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Part IIA Data sheet

Module 3C5 Dynamics

Module 3C6 Vibration

DYNAMICS IN THREE DIMENSIONS

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(\mathrm{e})}$$

where $\boldsymbol{p} = M\boldsymbol{v}_\mathrm{G}$, M is the total mass, $\boldsymbol{v}_\mathrm{G}$ is the velocity of the centre of mass and $\boldsymbol{F}^{(\mathrm{e})}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(\mathrm{e})} = (\boldsymbol{r}_\mathrm{G} - \boldsymbol{r}_\mathrm{P}) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_\mathrm{G}$$
$$= \dot{\boldsymbol{h}}_\mathrm{P} + \dot{\boldsymbol{r}}_\mathrm{P} \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(\mathrm{e})}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_\mathrm{P}$ and $\boldsymbol{h}_\mathrm{G}$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_\mathrm{P} = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_\mathrm{G} + (\boldsymbol{r}_\mathrm{G} - \boldsymbol{r}_\mathrm{P}) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_\mathrm{P} = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \mathbf{r})\,dm = \boldsymbol{I}\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $A = \int (y^2 + z^2)\,dm \qquad B = \int (z^2 + x^2)\,dm \qquad C = \int (x^2 + y^2)\,dm$

$D = \int yz\,dm \qquad E = \int zx\,dm \qquad F = \int xy\,dm$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(\mathrm{e})}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B - C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C - A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A - B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\ddot{\underline{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\dot{\underline{y}}^t M \dot{\underline{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M\underline{u}^{(n)} .$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\,\underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 6 for examples.

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 4 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 4) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\, u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f} \ .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \ge 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t\, e^{-\omega_n\zeta_n t}$$

for $t \ge 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \ge 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t\, e^{-\omega_n\zeta_n t}$$

for $t \ge 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \, e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

GOVERNING EQUATIONS FOR CONTINUOUS SYSTEMS

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)(a^4 - b^4)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.