EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 24 April 2019 9.30 to 11.10

Module 3C7

MECHANICS OF SOLIDS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C7 formulae sheet (2 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Under plane strain conditions, a strain field in Cartesian co-ordinates (x, y) is given by:

$$\begin{aligned} \varepsilon_{xx} &= A_1 x + A_2 xy + A_3 y^2 \\ \varepsilon_{yy} &= B_1 + B_2 x + B_3 y \\ \gamma_{xy} &= C_1 x + C_2 y + C_3 xy \end{aligned}$$

where A_i, B_i, C_i $(i = 1, 2, 3)$ are constants.

(a) Determine any constraints on A_i, B_i, C_i to ensure that this represents a valid strain field. [20%]

(b) Obtain expressions for the displacements $u(x, y)$ and $v(x, y)$ in the x and y directions, respectively. [50%]

(c) Discuss the physical interpretation of the unknown integration constants in (b). [15%]

(d) If the boundary conditions are defined so that the origin is fixed, and that no point along the x-axis can displace in the y-direction, which of A_i, B_i, C_i and the integration constants in (b) can be specified? [15%]

2 (a) Show that the equilibrium of a circular disc of variable thickness $h(r)$ subjected to plane stress axisymmetric loading satisfies

$$\frac{d(h\sigma_{rr})}{dr} = \frac{h}{r}(\sigma_{\theta\theta} - \sigma_{rr}),$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial and hoop stresses, respectively, at radius r. [25%]

(b) A variable thickness circular turbine blade of outer radius b is shrunk-fit onto a rigid shaft of radius a resulting in an interfacial pressure p as shown in the cross-sectional sketch in Fig. 1. The thickness of the turbine blade is $h = Ha/r$ where H is the thickness at the inner radius a. Show that the stress components

$$\sigma_{rr} = A - \frac{B}{r}, \qquad \sigma_{\theta\theta} = \frac{B}{r},$$

where A and B are constants, give an elastic solution. [15%]

(c) The turbine blade is made of a Tresca material with tensile yield strength Y. Calculate the interfacial pressure p at the initiation of yield in the disc, assuming plane stress conditions. [30%]

(d) Calculate an expression for the pressure at which the turbine disc becomes fully plastic by considering the solution of the equation in part (a) for the compound quantity $h\sigma_{rr}$. [30%]

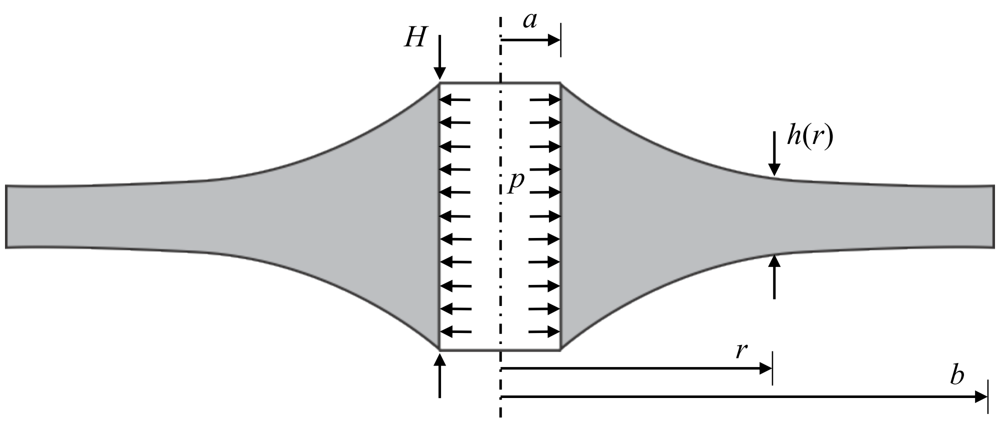

Fig. 1

3 A uniform horizontal beam of length L and rectangular cross-section (depth d and width b) is shown in Fig. 2. It is simply supported at $x = \pm L/2$. A downward load W is uniformly distributed on the upper surface of the beam and the lower surface is traction-free. The Airy stress function

$$\phi = \frac{W}{20Lbd^3}\left[20x^2y^3 - 15d^2x^2y - 4y^5 - 5d^3x^2 - y^3(5L^2 - 2d^2)\right]$$

is used to investigate the stresses within the beam.

(a) Confirm that this stress function gives stresses that satisfy the boundary conditions on the lower and upper surfaces of the beam. [30%]

(b) Show that these stresses satisfy the boundary conditions at the simply supported ends. [40%]

(c) What is the least value of the beam aspect ratio L/d, for the longitudinal tensile stress at $(x, y) = (0, d/2)$ to be within 1% of the value given by beam theory? [30%]

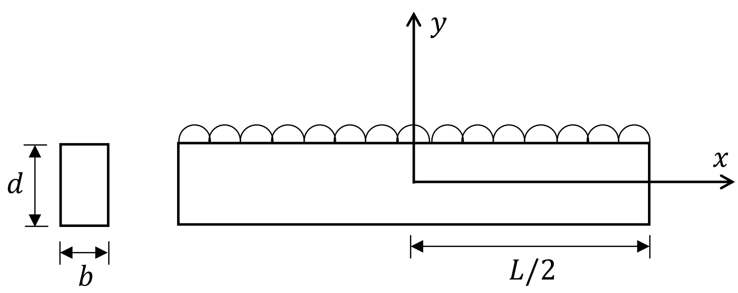

Fig. 2

4 Soil is loaded by a force F through a footing of width $2a$ in plane strain, as shown in Fig. 3. The soil is assumed to be isotropic, incompressible, rigid-perfectly plastic with a flow stress in shear of magnitude k.

(a) Explain the difference between the upper bound and lower bound theorems of plasticity. [20%]

(b) Assume that tangential velocity discontinuities occur along the thin lines indicated in Fig. 3. All triangular blocks shown are congruent isosceles triangles of fixed width a and arbitrary height b.

(i) Consider the case of a "smooth" interface where there is no friction between the footing and the soil. Derive an optimal expression for an upper bound on the force F. [40%]

(ii) Now consider the extreme case of a "rough" interface with sticking friction between the footing and the soil. Derive a revised optimal expression for an upper bound on the force F. [30%]

(iii) What are the consequences for design of the results from (i) and (ii)? [10%]

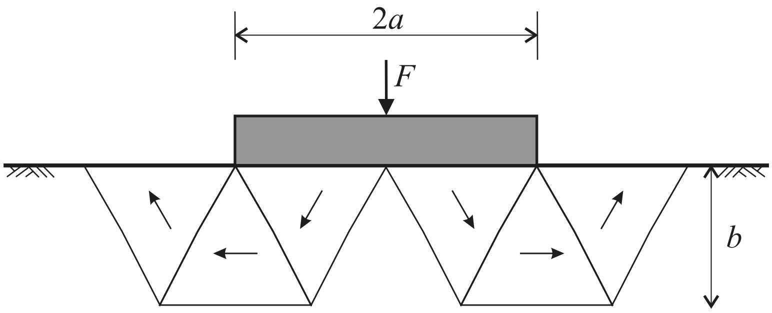

Fig. 3

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Module 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

Plane strain elastic constants $\quad \overline{E} = \dfrac{E}{1-\nu^2}\ ;\ \overline{\nu} = \dfrac{\nu}{1-\nu}\ ;\ \overline{\alpha} = \alpha(1+\nu)$

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity with no thermal strains or body forces)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (=\tau_x) = \dfrac{\partial\psi}{\partial y}$, $\sigma_{zy}\ (=\tau_y) = -\dfrac{\partial\psi}{\partial x}$

Equilibrium: $T = 2\int_A \psi\, dA$

Governing equation for elastic torsion: $\nabla^2\psi = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1\sigma_P^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\left\{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\right\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1-\sigma_2|$, $|\sigma_2-\sigma_3|$ or $|\sigma_3-\sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1-\sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2-\sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}.$$