EGT2
ENGINEERING TRIPOS PART IIA

Thursday 25 April 2019 2 to 3.40

Module 3D7

FINITE ELEMENT METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Engineering Data Book
Attachment: 3D7 datasheet (3 pages)
CUED approved calculator allowed

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 The equilibrium equation for an elastic bar is:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(EA\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f,$$

where E is the Young's modulus, A the cross-sectional area, u the displacement and f a distributed force. Consider the case in which E and A are constant, and where the bar extends from x_1 to x_2 ($x_1 < x_2$).

(a) Derive the weak formulation for a bar with an applied force h at x_1, and a prescribed displacement $u(x_2) = 0$. [20%]

(b) The bar is heated to a prescribed temperature $T(x)$, causing a thermal strain of $\epsilon_T = \alpha(T(x) - T_0)$, where α and T_0 are constants.

(i) Derive the weak formulation of the equilibrium equation for the bar for the case $u(x_1) = u(x_2) = 0$. [20%]

(ii) For the case $f = 0$ and $T(x)$ is constant, compute the right-hand side vector for a single linear element of length l. [20%]

(iii) If f is constant and $T(x)$ varies linearly, and the element right-hand side vector is computed using Gauss quadrature, how many quadrature points would you recommend? [10%]

(c) The bar problem is computed now using a quadratic isoparametric element. The nodes of the parent element are at $\zeta_1 = -1$, $\zeta_3 = 0$ and $\zeta_2 = 1$ and an element e has physical nodal positions $x_1^e < x_3^e < x_2^e$.

(i) For the point $\zeta = 0$ in the parent element, give its physical coordinate. [10%]

(ii) Give an expression for $\mathrm{d}u/\mathrm{d}x$ as a function of ζ, the nodal displacements u_i, and the nodal coordinates x_i^e. [20%]

2 A rail track is modelled as a beam on an elastic foundation, as shown in fig. 1. Each section of track has length L, and between each section is a small gap. The equilibrium equation for a static beam is

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}\right) = f,$$

where EI is the bending stiffness, v is the deflection and f is the force per unit length applied along the beam.

(a) Accounting for inertia effects, if the beam has density ρ per unit length and the foundation has stiffness k, derive the weak form for this problem in the absence of any train loading. [40%]

(b) What finite element type would you recommend for this problem, and why? [10%]

(c) If train loading is included, how might you add it to the finite element formulation? [10%]

(d) If the semi-discrete finite element formulation is expressed as:

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f},$$

formulate a problem using the scheme:

$$\begin{aligned} u_{n+1} &= u_n + \dot{u}_n \Delta t + \frac{1}{2}\ddot{u}_n \Delta t^2 \\ \dot{u}_{n+1} &= \dot{u}_n + \frac{1}{2}\left(\ddot{u}_n + \ddot{u}_{n+1}\right)\Delta t \end{aligned}$$

that can be solved. Would your problem require the solution of a linear system, and would you recommend this scheme? [40%]

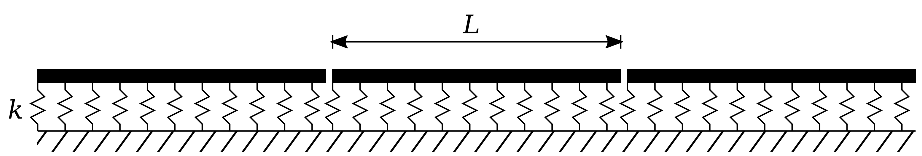

Fig. 1

3 Figure 2 shows an isoparametric element $QRST$ in the physical domain, and its parent element, a standard four-node quadrilateral, element $qrst$.

(a) Find the bilinear shape functions for the nodes of the parent element. [10%]

(b) Considering an isoparametric map, give the expression for the physical coordinates of a point in terms of its coordinates in the parent element. [25%]

(c) Calculate the coordinates of the point P in the physical element corresponding to point $p = (0.5, 0.5)$ in the parent element. [15%]

(d) For the straight lines rs and qps in the parent element, sketch the mapped lines on the physical element. Are the mapped lines straight? [25%]

(e) Find the displacement components at point P due to the nodal displacements:

$$\boldsymbol{u}_Q = (0.2, 0.1), \ \ \boldsymbol{u}_R = (0.0, 0.15), \ \ \boldsymbol{u}_S = (0.0, 0.0), \ \ \boldsymbol{u}_T = (-0.05, 0.0).$$

[25%]

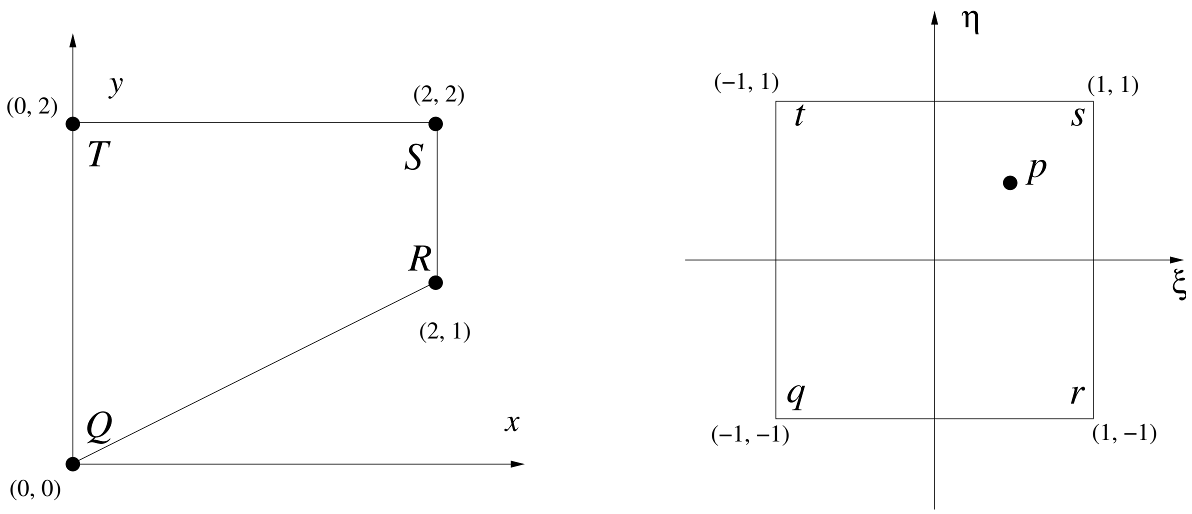

Fig. 2

4 Consider the heat equation on the rectangular domain shown in Fig. 3 (a). The outward unit normal vector to the boundary is denoted by $\boldsymbol{n}$. The heat flux is given by $\boldsymbol{q} = -\boldsymbol{D}\nabla T$, where $\boldsymbol{D}$ is the conductivity matrix and T is the temperature. A heat source s is present throughout the domain. On the top and bottom boundaries $\boldsymbol{q} \cdot \boldsymbol{n} = 0$, and on the right-hand side boundary $\boldsymbol{q} \cdot \boldsymbol{n} = f$. On the left-hand side boundary $T = 1$.

(a) State the strong form of the heat equation for this problem. [10%]

(b) Deduce the weak formulation for this problem. [20%]

(c) Let $s = 1$, $f = 1$ and $\boldsymbol{D} = 4\boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. We seek the finite element solution of this problem using two three-noded triangular elements, as shown in Fig. 3 (b). The node and element numbering is indicated in the figure.

(i) Calculate the $\boldsymbol{B}$ matrix for each element. [10%]

(ii) Calculate the element matrices

$$\int_{[1]} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} d\Omega \quad \text{and} \quad \int_{[2]} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} d\Omega.$$

[10%]

(iii) Calculate the element source vectors

$$\int_{[1]} s \boldsymbol{N}^T d\Omega \quad \text{and} \quad \int_{[2]} s \boldsymbol{N}^T d\Omega.$$

[10%]

(iv) Calculate the boundary flux vector for element [2]. [20%]

(v) Find the temperature T at nodes 2 and 3. [20%]

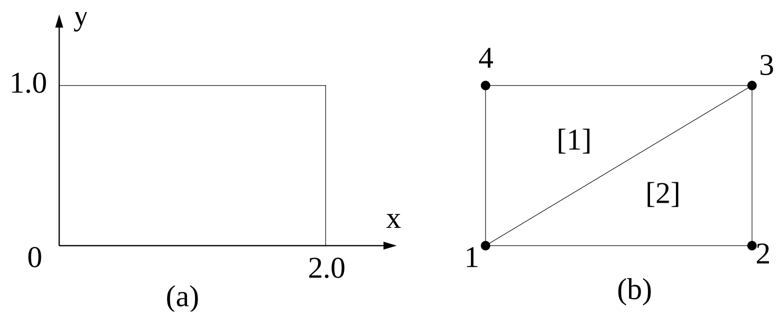

Fig. 3

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