EGT2
ENGINEERING TRIPOS PART IIA

Thursday 25 April 2019 9.30 to 11.10

Module 3F3

STATISTICAL SIGNAL PROCESSING

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Engineering Data Book
CUED approved calculator allowed

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 Let $X_0, X_1, \ldots$ be a Markov chain which takes values from the finite set $\{0, 1, \ldots, M\}$ and has the following transition probability matrix

$$\Pr(X_{n+1} = j | X_n = i) = \begin{cases} \alpha & \text{if } j = i + 1 \\ 1 - \alpha & \text{if } j = i - 1 \end{cases}$$

for $0 < i < M$. The transition probabilities for $i = 0$ or M are: $\Pr(X_{n+1} = 0 | X_n = 0) = 1 - \alpha$, $\Pr(X_{n+1} = 1 | X_n = 0) = \alpha$, $\Pr(X_{n+1} = M | X_n = M) = \alpha$, $\Pr(X_{n+1} = M - 1 | X_n = M) = 1 - \alpha$. Assume $X_0 = 0$.

(a) Show, for $0 < i < M$,

$$\Pr(X_{n+1} = i) = \alpha \Pr(X_n = i - 1) + (1 - \alpha) \Pr(X_n = i + 1) \tag{1}$$

and give the corresponding expressions for $\Pr(X_{n+1} = 0)$ and $\Pr(X_{n+1} = M)$. [30%]

(b) Show that the Markov chain is irreducible. Find its stationary probability mass function when $\alpha = 0.5$. [20%]

(c) Let $x_1, \ldots, x_T$ be a sequence of states from the Markov chain. In this data set, let r be the number of self-transitions to state M, i.e. $x_{n-1} = x_n = M$ and let s be the number transitions that correspond to an increase in value, i.e. $x_{n-1} < x_n$. Find $\log \Pr(X_1 = x_1, \ldots, X_T = x_T | X_0 = 0)$ and the maximum likelihood estimate of α. [20%]

(d) The Markov chain can be used to model the number of customers in a queue. Let X_n denote the number of customers in the queue at time n and assume that there is no upper limit on the length of the queue, i.e. $M = \infty$.

(i) Show that $\sum_{i=1}^{\infty} i \ \Pr(X_n - 1 = i) \geq \mathbf{E}(X_n - 1)$. [10%]

(ii) Using equation (1) or otherwise, find an inequality relating $\mathbf{E}(X_{n+1})$ and $\mathbf{E}(X_n)$ and explain the behaviour of the queue length over time when $\alpha > 0.5$. [20%]

2 Let X_n follow an equation described by a linear trend in noise,

$$X_n = a + bn + W_n$$

where $\{W_n\}$ is a zero mean wide-sense stationary random process and a, b are unknown constants. Let $x_0, x_1, \ldots, x_{N-1}$ be a batch of N of data points.

(a) Find the least squares estimate $\hat{\theta} = (\hat{a}, \hat{b})^{\mathrm{T}}$ of the unknown parameters $\theta = (a, b)^{\mathrm{T}}$ and show that this least squares estimate is unbiased. (The superscript T denotes transpose.) [30%]

(b) Find the variance $\mathbf{E}(\hat{\theta}\hat{\theta}^{\mathrm{T}}) - \theta\theta^{\mathrm{T}}$. [20%]

(c) Show that the variance tends to 0 as N increases. (You may assume W_n is white noise.) [10%]

(d) The model can be used to describe a data set with a linear trend and a sinusoidal component. Let $W_n = c \sin(\omega_0 n + \phi)$ where ϕ is a Uniform random variable with range $(-\pi, \pi)$,

(i) Show W_n is wide-sense stationary and find its mean and autocorrelation function. [20%]

(ii) Find the power spectral density of W_n. [10%]

(iii) How might an estimate of ω_0 be found from the data? [10%]

3 A Gaussian random variable X with mean μ and variance σ^2 is observed in Gaussian noise,

$$Y_n = X + W_n$$

for $n = 1, \ldots, N$ where W_n are independent zero mean Gaussian random variables with variance σ_w^2.

(a) Find the joint probability density function (pdf) $p(x, y_1, \ldots, y_n)$ of X and $Y_1, \ldots, Y_n$. [10%]

(b) Show that the conditional pdf $p(x|y_1)$ of X given $Y_1 = y_1$ is Gaussian with

$$\text{mean } \mu_1 = \frac{\sigma_w^2 \mu + \sigma^2 y_1}{\sigma^2 + \sigma_w^2} \qquad \text{and} \qquad \text{variance } \sigma_1^2 = \frac{\sigma_w^2 \sigma^2}{\sigma^2 + \sigma_w^2}.$$

[30%]

(c) Show that

$$p(x|y_1, y_2) = \frac{p(y_2|x)p(x|y_1)}{p(y_2|y_1)}$$

and find the means μ_k and variances σ_k^2 of the conditional pdfs of X after observing $y_1, \ldots, y_k$ for $k = 1, \ldots, n$. [20%]

(d) Let

$$Z = \frac{1}{n}(Y_1 + Y_2 + \ldots + Y_n) = X + V.$$

V is a Gaussian random variable. Find the mean and variance of V and then find the conditional pdf of X given $Z = z$. [20%]

(e) Let $z = (y_1 + \ldots + y_n)/n$. Compare the means and variances of $p(x|y_1, \ldots, y_n)$ and $p(x|z)$ to decide which scheme is a more efficient use of the available data. [20%]

4 Let X_n be the random process

$$X_n = \alpha X_{n-1} + W_n$$

where α is a constant and $\{W_n\}$ is zero mean white noise, i.e. $\mathbf{E}(W_n) = 0$, $\mathbf{E}(W_n^2) = \sigma^2$ and $\mathbf{E}(W_n W_{n+k}) = 0$ for $k \neq 0$.

(a) Find the autocorrelation function $R_X(k) = \mathbf{E}(X_n X_{n+k})$ for all $k \geq 0$. [20%]

(b) Let

$$U_n = X_n + V_n$$

where V_n is zero mean white noise with $\mathbf{E}(V_n^2) = \sigma_v^2$. Assume V_n is uncorrelated with the random process W_n, that is $\mathbf{E}(W_n V_m) = 0$ for all m, n. Find the autocorrelation function of U_n. [15%]

(c) Find the best linear predictor $\hat{X}_n$ of X_n of the form

$$\hat{X}_n = h_1 U_{n-1}$$

by minimising the mean square error (MSE) $\mathbf{E}\left[(X_n - \hat{X}_n)^2\right]$ with respect to h_1. [15%]

(d) A different estimate of X_n is obtained by using the following scheme

$$\hat{X}_n = h_1 U_{n-1} + \ldots + h_p U_{n-p}$$

Find the values of $h_1, \ldots, h_p$ that minimise the MSE. [30%]

(e) When $\sigma_v^2 = 0$, find the optimal value of p. [20%]

END OF PAPER

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