Version GC/4

EGT2
ENGINEERING TRIPOS PART IIA

Thursday 2 May 2019   9:30 to 11:10

Module 3M1

MATHEMATICAL METHODS

*Answer **all three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) For a matrix $\boldsymbol{A} \in \mathbb{C}^{m \times n}$, if $\boldsymbol{A} = \boldsymbol{A}^H = \boldsymbol{A}^T$, what can you say about $\boldsymbol{A}$? [5%]

(b) If $\boldsymbol{Q} \in \mathbb{C}^{n \times n}$ is a unitary matrix, then $\boldsymbol{Q}^H \boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^H = \boldsymbol{I}$.

(i) What can be said about the determinant of $\boldsymbol{Q}$? [10%]

(ii) Show that $\|\boldsymbol{x}\|_2 = \|\boldsymbol{Q}\boldsymbol{x}\|_2$ for $\boldsymbol{x} \in \mathbb{C}^n$. [10%]

(iii) Show that $\|\boldsymbol{A}\|_2 = \|\boldsymbol{Q}\boldsymbol{A}\|_2 = \|\boldsymbol{A}\boldsymbol{Q}\|_2$ for $\boldsymbol{A} \in \mathbb{C}^{n \times n}$. [20%]

(iv) Would you expect the l_1 and l_∞ norms of $\boldsymbol{x}$ and $\boldsymbol{Q}\boldsymbol{x}$ to be the same? [5%]

(c) For a Hermitian matrix $\boldsymbol{M} \in \mathbb{C}^{n \times n}$ and vector $\boldsymbol{x} \in \mathbb{C}^n$, the Rayleigh quotient is defined as:

$$R(\boldsymbol{M}, \boldsymbol{x}) = \frac{\boldsymbol{x}^H \boldsymbol{M} \boldsymbol{x}}{\boldsymbol{x}^H \boldsymbol{x}}.$$

(i) Prove that the Rayleigh quotient is real valued. [10%]

(ii) Show that $R(\boldsymbol{M}, \boldsymbol{x}) = R(\boldsymbol{M}, c\boldsymbol{x})$, where $c \neq 0$ is a scalar. [10%]

(iii) If $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of $\boldsymbol{M}$, respectively, prove that:

$$\lambda_{\min} \leq R(\boldsymbol{M}, \boldsymbol{x}) \leq \lambda_{\max}$$

for any $\boldsymbol{x} \in \mathbb{C}^n$. [20%]

(iv) What is $\boldsymbol{x}$ when $R(\boldsymbol{M}, \boldsymbol{x})$ is maximum? [10%]

2 Consider the following optimisation problem,

$$\text{minimize} \quad x^2 + 2y^2$$

subject to the constraint

$$x + 2y = 3.$$

(a) Draw the contour lines of the objective function and the equation of the constraint. [20%]

(b) (i) Introduce a penalty with strength μ for the violation of the constraint, and find the solution of the penalised problem for a given μ. [25%]

(ii) Draw the locus of solutions on your diagram in part (a) for different values of the penalty strength. [20%]

(c) (i) Solve the original problem using the Lagrange multiplier method. [25%]

(ii) What is the geometric interpretation of the value of the Lagrange multiplier? [10%]

3 Figure 1 shows the state-diagram for a process. The aim is to generate samples from this process when the process has a particular *stationary distribution*. The transition matrix associated with this process is **P** and the values of the scalars a and b are to be set.

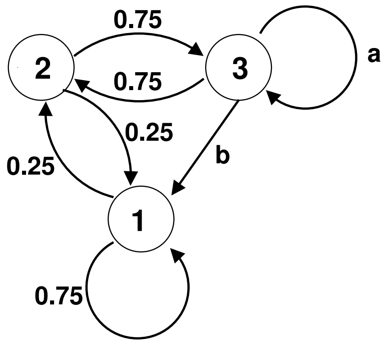

Fig. 1

(a) What is the state transition matrix, **P**, associated with the state transition diagram in Fig 1? What constraints should be satisfied by a and b? [15%]

(b) The transition probabilities a and b are to be set so that the stationary distribution, $\boldsymbol{\pi}$, associated with **P** is

$$\boldsymbol{\pi} = \left[\begin{array}{ccc} 1/3 & 1/3 & 1/3 \end{array} \right]$$

What are the values of a and b that result in this stationary distribution? [20%]

(c) A Markov process with stationary distribution $\boldsymbol{\pi}$ and transition matrix **P** is said to satisfy detailed balance when the following equation is satisfied for all pairs of states, j and k, of the process:

$$\pi_j p_{j,k} = \pi_k p_{k,j}$$

(i) Show that process described in Fig 1, with the values of a and b computed in Part (b), satisfies detailed balance. [10%]

(ii) Show that any distribution $\boldsymbol{\pi}$ that satisfies detailed balance is a stationary distribution of the transition matrix **P**. [10%]

(d) Metropolis-Hastings is to be used to generate samples that have the stationary distribution $\boldsymbol{\pi}$ in Part (b). A proposal process, with transition matrix $\mathbf{R}$, for generating samples is to be used where $r_{j,j} = 0$ for all states. The probability of accepting a generated sample from the proposal process, $\hat{X}_{i+1}$, is given by α where

$$\alpha = \min\left\{\frac{\pi_{\hat{X}_{i+1}} r_{\hat{X}_{i+1},X_i}}{\pi_{X_i} r_{X_i,\hat{X}_{i+1}}}, 1\right\}$$

where X_i is the state at time instance i.

(i) Show that the equivalent transition matrix for this process, $\overline{\mathbf{R}}$ has terms

$$\bar{r}_{j,k} = \begin{cases} r_{j,k}\min\left\{\frac{\pi_k r_{k,j}}{\pi_j r_{j,k}}, 1\right\} & \text{if } j \neq k; \\ 1 - \sum_{k\neq j} r_{j,k}\min\left\{\frac{\pi_k r_{k,j}}{\pi_j r_{j,k}}, 1\right\} & j = k \end{cases}$$

[15%]

(ii) Using Part (c), or otherwise, show that $\boldsymbol{\pi}$ is a stationary distribution of this process. [20%]

(iii) Will this process generate samples from the process with transition matrix $\mathbf{P}$ with the values of a and b given in Part (b)? Justify your answer. [10%]

END OF PAPER

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