Paper 4: Mathematical Methods

Solutions to 2018 Tripos Paper

1. **Complex equation**

   Make the substitution $u = z^i$ to obtain $u^2 - 3u + 2 = 0$ with solutions $u = (3 \pm \sqrt{9-8})/2 \Rightarrow u = 1$ or $u = 2$. Then use $u = z^i \Rightarrow z = e^{-i\log(u)}$ and plug into Euler's identity to get the two solutions $\underline{z = 1 + i0}$ and $\underline{z = \cos(\log(2)) - i\sin(\log(2))}$. [10]

2. **Limit**

   The fraction $\frac{x}{y} \to 0$ as $y \to \infty$, plugging into the Taylor series for $\log(1 + \frac{x}{y}) \simeq \frac{x}{y} + O((\frac{x}{y})^2)$, therefore $\underline{y\log(1 + \frac{x}{y}) \to x}$ as $y \to \infty$. [10]

3. **Difference equation**

   Set $x_n = \lambda^n$ to obtain $\lambda^2 + \lambda - 6 = 0$ with solutions $\lambda = (-1 \pm \sqrt{1+24})/2$ or $\lambda = -3, 2$. Therefore the general solution is $x_n = a2^n + b(-3)^n$. Boundary condition $x_0 = x_1 \Rightarrow a + b = 2a - 3b \Rightarrow a = 4b$, so the particular solution is $\underline{x_n = b(2^{n+2} + (-3)^n)}$. [10]

4. **Vectors**

   (a) Since $\mathbf{m}$ and $\mathbf{n}$ are orthogonal, and both $(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{n} = 0$ and $(\mathbf{m} \times \mathbf{n}) \cdot \mathbf{m} = 0$, it follows that $\underline{\mathbf{m}, \mathbf{n} \text{ and } \mathbf{m} \times \mathbf{n} \text{ are orthogonal}}$ (and thus span the entire space). [5]

   (b)

   $$\mathbf{x} \;=\; \alpha\mathbf{m} + \beta\mathbf{n} + \gamma\,\mathbf{m} \times \mathbf{n} \;\Rightarrow \begin{cases} \mathbf{x} \cdot \mathbf{m} = \alpha \\ \mathbf{x} \cdot \mathbf{n} = \beta \\ \mathbf{x} \cdot \mathbf{x} = \alpha^2 + \beta^2 + \gamma^2, \end{cases}$$

   which we substitute back into eq. (1), yielding

   $$4\mathbf{x} \cdot \mathbf{m} = \mathbf{x} \cdot \mathbf{x} - (\mathbf{x} \cdot \mathbf{m})^2 - (\mathbf{x} \cdot \mathbf{n})^2 \;\Rightarrow\; 4\alpha = \alpha^2 + \beta^2 + \gamma^2 - \alpha^2 - \beta^2 \;\Rightarrow\; \underline{4\alpha = \gamma^2}.$$

   $\underline{\text{Any value of } \beta}$. Geometric interpretation: a parabola in the $\mathbf{m}$ direction as a function of $\mathbf{m} \times \mathbf{n}$ direction, for any position along the $\mathbf{n}$ direction, ie parabolic cylinder. [10]

   (c) (i) The plane $\mathbf{r} \cdot \mathbf{m} = -1$, with normal $\mathbf{m}$ and $-\mathbf{m}$ on the plane; therefore the shortest distance is in the (normal) $\mathbf{m}$ direction, thus $(\mathbf{x} - (-\mathbf{m})) \cdot \mathbf{m} = \underline{\alpha + 1}$.

   (ii) Shortest distance from $\mathbf{x}$ to line $\mathbf{r} = \mathbf{m} + \lambda\mathbf{n}$ is given by

   $$\begin{aligned} \ell \;&=\; (\mathbf{x} - \mathbf{m}) \times \mathbf{n} \;=\; [(\alpha - 1)\mathbf{m} + \beta\mathbf{n} + \gamma\,\mathbf{m} \times \mathbf{n}] \times \mathbf{n} \\ &=\; (\alpha - 1)\mathbf{m} \times \mathbf{n} + \gamma((\mathbf{n} \cdot \mathbf{m})\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})\mathbf{m}) \;=\; \sqrt{(\alpha - 1)^2 + \gamma^2}. \end{aligned}$$

   Plugging in $\gamma^2 = 4\alpha$, yields $\ell = \sqrt{(\alpha - 1)^2 + 4\alpha} = \underline{1 + \alpha}$. The two distances are equal. [15]

5. **Linear algebra**

(a) $\det(A - \lambda I) = (1-\lambda)(2-\lambda)(3-\lambda) - a^2(2-\lambda) = (2-\lambda)(\lambda^2 - 4\lambda + 3 - a^2 = 0$ with solutions $\underline{\lambda = 2}$ and $\lambda = (4 \pm \sqrt{16 - 12 + 4a^2})/2 = \underline{2 \pm \sqrt{1 + a^2}}$. [10]

(b) Middle eigenvalue $\lambda = 2$, thus we seek $\mathbf{x}$ such that $A\mathbf{x} = 2\mathbf{x}$, which gives rise to $x_1 + ax_3 = 2x_1$ and $ax_1 + 3x_3 = 2x_3$ (as well as the trivial $2x_2 = 2x_2$). The two first equations only have the solution $x_1 = x_3 = 0$, so the normalized eigenvector must be $\underline{\mathbf{x} = (0, \pm 1, 0)^\top}$. [10]

(c) Eigenvalues of A^{-1} are the reciprocal of those of A, ie $\underline{\lambda = \frac{1}{2}}$ and $\underline{\lambda = (2 \pm \sqrt{1 + a^2})^{-1}}$. [5]

(d) Eigenvalues of A^3 are the third power of eigenvalues of A, thus $\underline{\det(A^3) = 8(3 - a^2)^3}$. [5]

Paper 4: Mathematical Methods

Solutions to 2018 Section B

6. **Laplace transforms**

Taking Laplace transforms of both sides of the differential equation:

$$X(s^2 + 2s + 5) = X\left[(s+1)^2 + 4\right] = \frac{1}{s}$$

$$\begin{aligned} \text{Hence } X &= \frac{1}{s\left[(s+1)^2+4\right]} = \frac{A}{s} + \frac{B}{\left[(s+1)^2+4\right]} + \frac{C(s+1)}{\left[(s+1)^2+4\right]} \\ &= \frac{A\left[(s+1)^2+4\right] + Bs + Cs(s+1)}{s\left[(s+1)^2+4\right]} = \frac{s^2(A+C) + s(2A+B+C) + 5A}{s\left[(s+1)^2+4\right]} \end{aligned}$$

By inspection, $A = 1/5, B = C = -1/5$, and therefore

$$\begin{aligned} X(s) &= \frac{1}{5}\left[\frac{1}{s} - \frac{1}{(s+1)^2+4} - \frac{(s+1)}{(s+1)^2+4}\right] \\ x(t) &= \frac{1}{5}\left[1 - e^{-t}\left(\frac{1}{2}\sin 2t + \cos 2t\right)\right] \quad \text{for } t \geq 0 \end{aligned}$$ [10]

Assessors' remarks: This question was answered very well by almost all candidates, though some made life difficult for themselves by not adopting the suggested partial fraction decomposition from the outset.

7. **Linear systems and convolution**

We need to evaluate the following convolution integral:

$$\begin{aligned} y(t) &= \int_0^t x(\tau)g(t-\tau)\,d\tau = \int_0^t e^{-\tau}\left(e^{-(t-\tau)} - e^{-2(t-\tau)}\right)\,d\tau \\ &= \int_0^t \left(e^{-t} - e^{-2t}e^{\tau}\right)\,d\tau = \left[\tau e^{-t} - e^{-2t}e^{\tau}\right]_0^t = te^{-t} - e^{-t} + e^{-2t} \end{aligned}$$

The response to the input is therefore

$$y(t) = \begin{cases} 0 & t < 0 \\ te^{-t} - e^{-t} + e^{-2t} & t \geq 0 \end{cases}$$

The impulse response is continuous at $t = 0$: this is what we would expect from a second order system. First order systems have impulse responses that are discontinuous at $t = 0$. [10]

Assessors' remarks: Almost every candidate knew to use convolution, though many failed to evaluate the integral accurately. Justifications for whether this was a first or second order system were not always sound.

8. **Probability**

For A to win the match, the set score might be 3-0, 3-1 or 3-2 in favour of A.

3-0 There is only one way this can happen, with probability a^3

3-1 There are ${}^3C_1 = 3$ ways this can happen (B winning sets 1, 2 or 3), so the probability is $3(1-a)a^3$

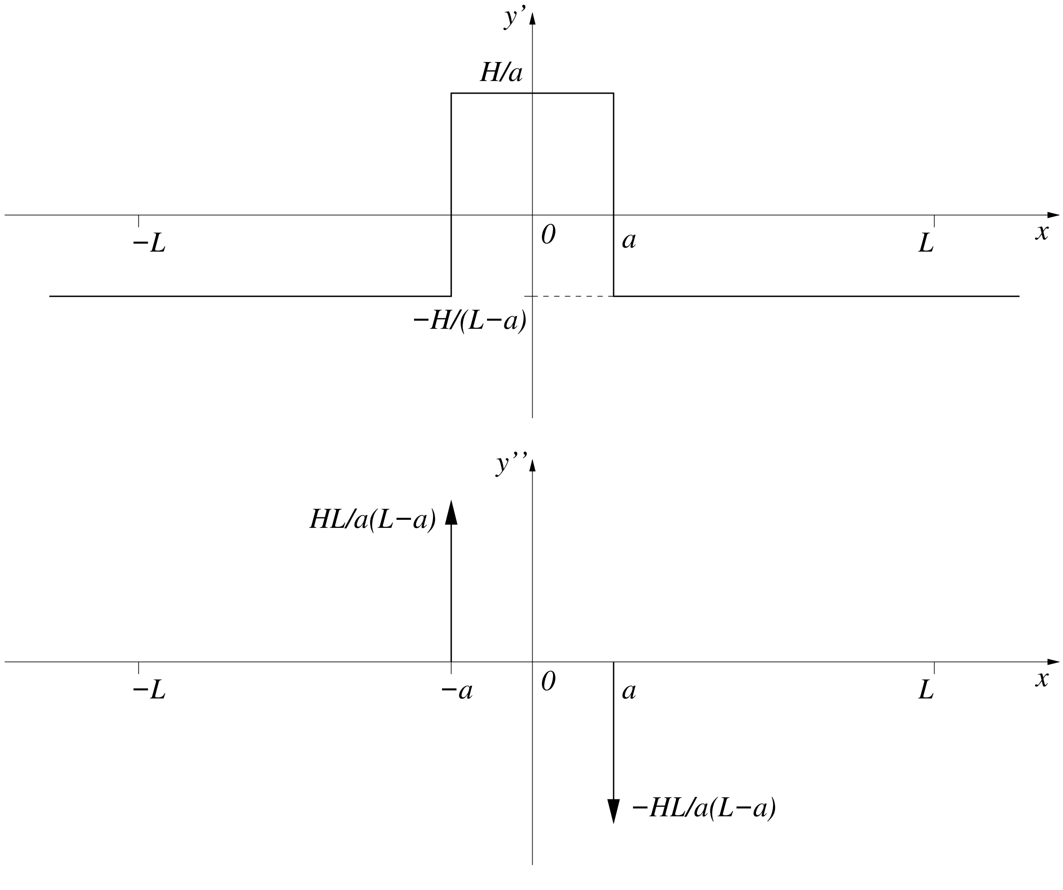
3-2 There are ${}^4C_2 = 6$ ways this can happen (B winning any 2 sets out of the first 4), so the probability is $6(1-a)^2a^3$

Hence $P = a^3 + 3(1-a)a^3 + 6(1-a)^2a^3 = a^3(10 - 15a + 6a^2)$. As expected, when $a = 0$, $P = 0$; when $a = 0.5$, $P = 0.5$; and when $a = 1$, $P = 1$. [10]

Assessors' remarks: This question was generally well answered, though many candidates exhaustively enumerated all the permutations, often by way of huge tree diagrams, instead of distilling the argument to a few, simple calculations.

9. **Fourier series**

(a) The first and second derivatives of $y(x)$ are sketched below.



$y''(x)$ is odd with period $2L$, and can therefore be expressed as a Fourier series as follows:

$$y''(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} y''(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{2}{L}\int_{0}^{L} y''(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = \frac{-2H}{a(L-a)}\sin\left(\frac{n\pi a}{L}\right)$$

by the sifting property of delta functions. Hence

$$y''(x) = \sum_{n=1}^{\infty} \frac{-2H}{a(L-a)} \sin\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

Integrating twice, we obtain

$$y(x) = \sum_{n=1}^{\infty} \frac{2HL^2}{n^2\pi^2 a(L-a)} \sin\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

where both constants of integration are zero, since $y(x)$ is periodic and has zero mean. [15]

(b) Considering only the Fourier coefficients, for concision:

$$\lim_{a\to 0} \frac{2HL^2}{n^2\pi^2 a(L-a)} \sin\left(\frac{n\pi a}{L}\right) = \lim_{a\to 0} \frac{2HL}{n^2\pi^2 a}\left(1-\frac{a}{L}\right)^{-1} \sin\left(\frac{n\pi a}{L}\right)$$
$$= \lim_{a\to 0} \frac{2HL}{n^2\pi^2 a}\left(1+\frac{a}{L}+\mathcal{O}(a^2)\right)\left(\frac{n\pi a}{L}+\mathcal{O}(a^2)\right) = \lim_{a\to 0} \frac{2HL}{n^2\pi^2 a}\left(\frac{n\pi a}{L}+\mathcal{O}(a^2)\right)$$
$$= \frac{2H}{n\pi}$$

Hence

$$y(x) = \frac{2H}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$$

Alternatively, the *Mathematics Data Book* provides the following Fourier series for a similar, but not identical, saw-tooth wave of period 2L:

$$y(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

To match the waveform in the question, the saw-tooth needs negating and scaling by H

$$y(x) = \frac{2H}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

and also shifting by half a period

$$\begin{aligned} y(x) &= \frac{2H}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} \sin\left(\frac{n\pi(x-L)}{L}\right) \\ &= \frac{2H}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} \left[\sin\left(\frac{n\pi x}{L}\right)\cos(n\pi) + \cos\left(\frac{n\pi x}{L}\right)\sin(n\pi)\right] \\ &= \frac{2H}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} \sin\left(\frac{n\pi x}{L}\right)(-1)^n = \frac{2H}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$ [10]

(c) When $a \neq 0$, $y(x)$ has a discontinuity in its gradient and we would therefore expect its Fourier series to converge as $1/n^2$, which is indeed what we find in (a). As $a \to 0$, the discontinuity in gradient becomes a discontinuity in value, and we would expect $1/n$ convergence, which is indeed what we find in (b). [5]

Assessors' remarks: A problematic question, in that very few candidates tackled (a) by first differentiating the function, either once (to a stepped function) or twice (to delta functions), despite this approach being encouraged in the corresponding examples paper. Those who differentiated did well. Some of those who embarked on the alternative, lengthy integration-by-parts succeeded in deriving the correct expression. However, most of the integration-by-parts attempts were flawed, and it was disappointing to see many candidates knowingly hide the flaws and fudge their way to the supplied formula. In (b), most candidates who took the limit did well, though there was lack of rigour (missing $O()$ terms) when using power series expansions. Attempts at manipulating the data book expression for the saw-tooth wave were less successful. In (c), the vast majority of candidates demonstrated a good understanding of the relationship between the discontinuities in a function and the rate of convergence of the corresponding Fourier series.

10. **Functions of three variables, contour plots, gradients**

(a)

$$\nabla T = \begin{bmatrix} \partial T/\partial x \\ \partial T/\partial y \\ \partial T/\partial z \end{bmatrix} = \cos(xy + x + y + 1 - z) \begin{bmatrix} y+1 \\ x+1 \\ -1 \end{bmatrix}$$

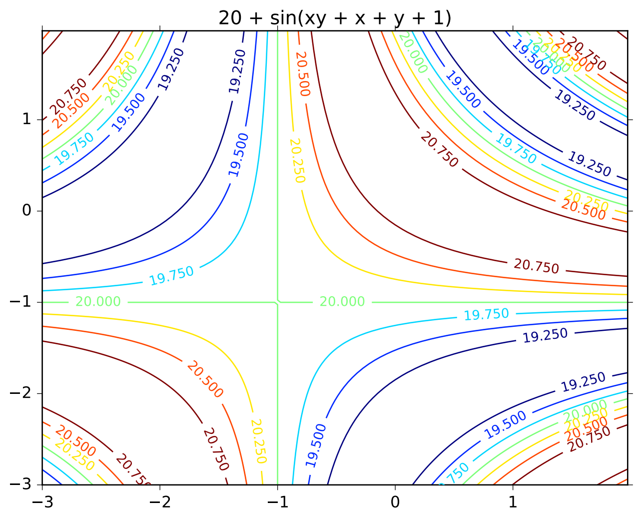
The units of ∇T are °C/m. [4]

(b) At the point $(1, 0, 2)$, $\nabla T = [\ 1\ 2\ \ -1\ ]^T$ and this is the normal to the tangent plane. The equation of the tangent plane is therefore $x + 2y - z = d$. Since the plane passes through the point $(1, 0, 2)$, it follows that $d = -1$. The equation of the tangent plane is therefore $x + 2y - z + 1 = 0$. [6]

(c) When $z = 0$ we have

$$T = 20 + \sin(xy + x + y + 1) = 20 + \sin W$$

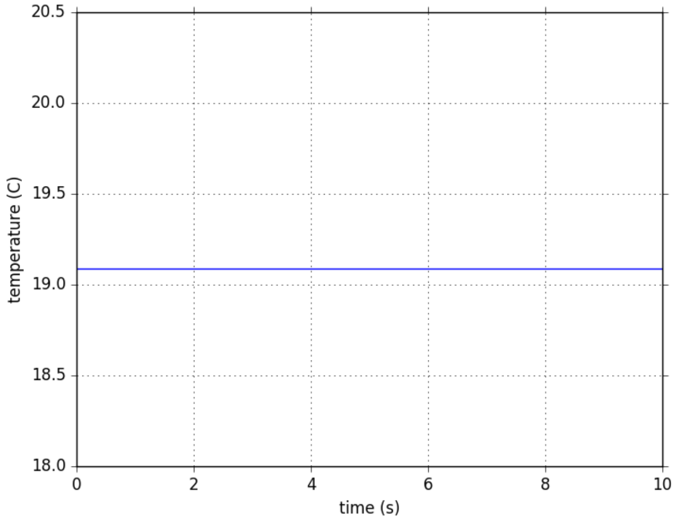
where $W = (x + 1)(y + 1)$. The important thing to realise is that contours of constant T are coincident with contours of constant W. It is clear that $W = 0$ when $x = -1$ or $y = -1$, giving two straight-line contours to get us started. The other contours are a smooth extrapolation of these, given that W is smooth and continuous.



[6]

(d) The bee's velocity is given by the differential of its position with respect to time, which is $(-2\sin 2t,\ 2\cos 2t,\ 2/\pi)$ m/s. As it passes through the point $(1, 0, 2)$ at time $t = \pi$ s, the bee's velocity is therefore $(0, 2, 2/\pi)$ m/s. We have already established that the temperature gradient at this point is $(1, 2, -1)$ °C/m. The rate of change of temperature experienced by the bee is therefore $[\ 0\ 2\ 2/\pi\ ]^T \cdot [\ 1\ 2\ -1\ ]^T = 4 - 2/\pi = 3.36$ °C/s. Alternatively, noting that the bee's speed is $\sqrt{4 + 4/\pi^2}$ m/s, the rate of change of temperature can be expressed as $(4 - 2/\pi)/\sqrt{4 + 4/\pi^2} = 1.60$ °C/m. [8]

(e) The wasp is flying along the line $y = 0, z = x + 3$. Substituting into the expression for T, we find $T = 20 + \sin[x + 1 - (x + 3)] = 20 + \sin(-2) = 19.1$ °C.



[6]

Assessors' remarks: There were many token attempts at this question, most likely a consequence of candidates committing too much time to the unnecessary integration-by-parts in the previous question. For the serious attempts, (a) was mostly well answered, though some candidates provided a scalar expression for ∇T. Solutions to (b) were variable, with many perfect answers but others not knowing where to start (despite the similar examples paper question) and others making careless slips. Part (c) was well answered, with most candidates identifying the $x = -1$ and $y = -1$ asymptotes, and sketching some plausible contours around them. Understanding of (d) was poor: full marks were awarded for answers expressed as either a rate of change with distance or a rate of change with time, as long as the candidates made it clear which one they were calculating, but in many cases there was no such clarity. Finally, most candidates correctly deduced that the wasp was flying along a line of constant temperature, but some thought it sufficient to demonstrate that $\nabla T = \mathbf{0}$ at only one point along the trajectory.

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June 2018

Engineering Tripos Part IA FIRST YEAR

Paper 4: Mathematical Methods

Solutions to 2018 Section C

11. **Python data structure**

(a) A Python `dictionary` is used with book titles as `keys` and rating as `values`. We can easily query the dictionary by key to get direct access to a specific value, i.e. we can find out the rating of a book based on its title without iterating through the list to check each name. [2]

(b) Determine three best rated childrens books. Output:
`['Gruffalo', 'Peppa Pig', 'My Little Dog']` [6]

(c) All audiobooks are sorted by popularity. [2]

Assessors' remarks: A popular question that was answered very well by almost all candidates. Some candidates did not read the question carefully and failed to specify the output of the print statement.

12. **Python class**

(a) Every object of type `BankAccount` has two attributes: `balance` and `account_holder`. Two methods are associated with this class, the method `credit`, which allows to increase the `balance` by a specified amount, and `__repr__`, which is called behind the scenes when using `print` in order to display `account_holder` followed by `balance`. [2]

(b) Add method `debit`

```
def debit(self, amount):
    if (amount > self.balance):
        print "Not sufficient funds"
    else:
        self.balance = self.balance - amount
```

[5]

(c) Output

```
    James Smith: 900
    950
    Not sufficient funds
    James Smith: 950
```

[3]

Assessors' remarks: The question was generally well answered. Some students found it difficult to identify class attributes and methods, and not everyone was careful specifying the output produced by the code.

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