ENGINEERING TRIPOS PART IA 2014

Paper 2 Structures and Materials

Solutions

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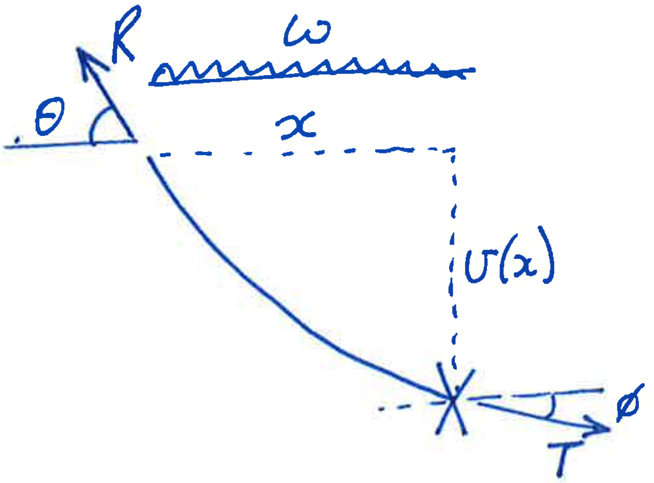
Ia tripos 2014 Paper 2: structures

1. (a) To ensure vertical equilibrium for the whole cable:

$$2R\sin\theta = 2L\omega \quad \therefore \quad R = \frac{\omega L}{\sin\theta}$$

(b) Taking a cut at an arbitrary position x:



The vertical deflection $v(x)$, the tension T and slope ϕ of the cable are all unknown. Taking moments about the cut eliminates T, ϕ.

Moments about the cut:

$$R\sin\theta . x = \frac{\omega x^2}{2} + R\cos\theta . v(x)$$

substituting the answer from (a) and rearranging,

$$\frac{\omega L}{\tan\theta} . v(x) = \omega L x - \frac{\omega x^2}{2}$$

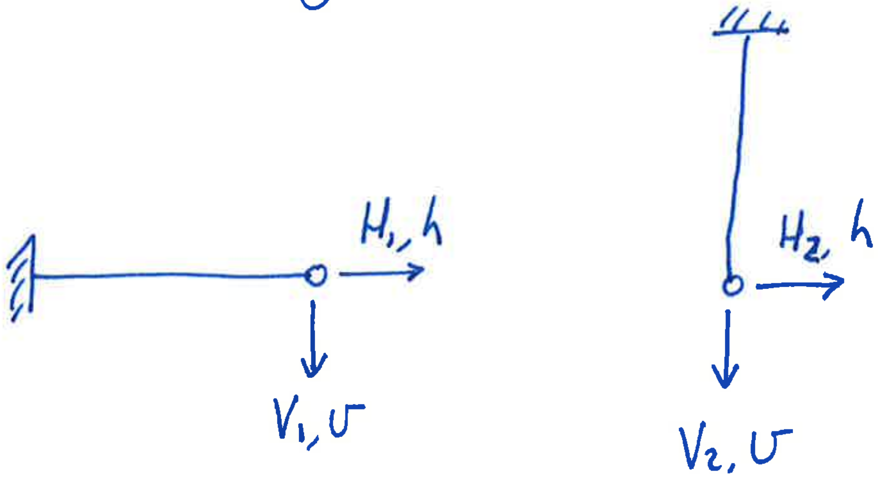
$$\therefore \quad v(x) = x\tan\theta\left(1 - \frac{x}{2L}\right)$$

2. (a) $E = \dfrac{\sigma}{\varepsilon}$ $\therefore$ $\dfrac{t}{L} . E = \dfrac{T}{A}$ $\therefore$ $t = \dfrac{LT}{AE}$

from the data book, $s = \dfrac{SL^3}{3EI}$

(b) Rearranging (a), $T = \dfrac{EAt}{L}$, $S = \dfrac{3EIs}{L^3}$

The loads H, V are split between the two struts, but both must deflect by h, v. So drawing them separately:



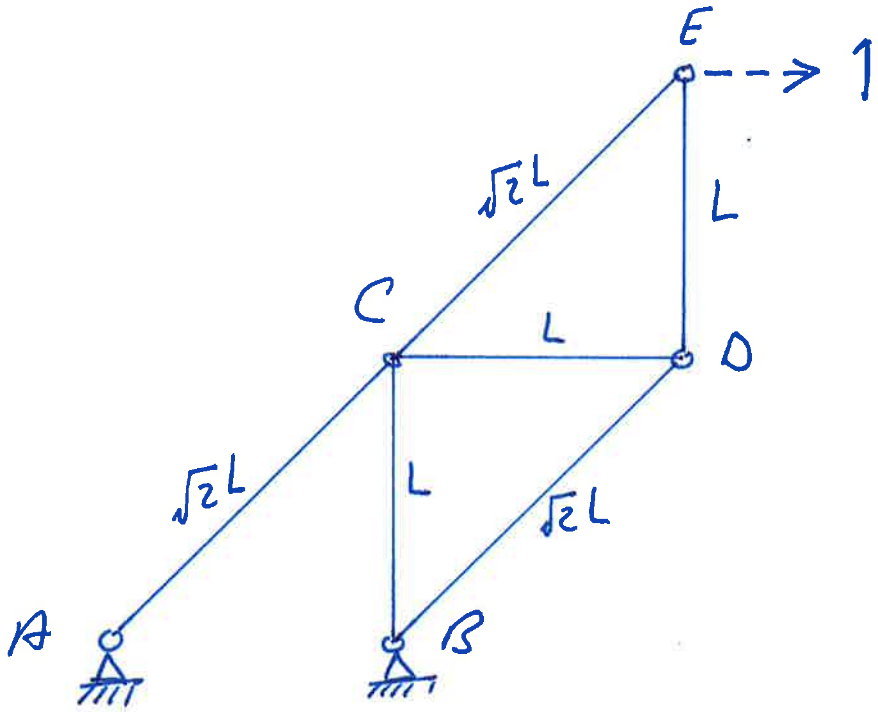
By equilibrium, $H = H_1 + H_2$

$$= \frac{EAh}{L} + \frac{3EI\,h}{L^3}$$

$$\therefore h = \frac{L^3}{E} \cdot \frac{H}{(AL^2 + 3I)}$$

similarly, $v = \dfrac{L^3}{E} \cdot \dfrac{V}{(AL^2 + 3I)}$

3 (a)



(b)

Member	Extension ($\times \frac{L}{AE}$)	Length ($\times L$)	Force
AC	4	$\sqrt{2}$	$2\sqrt{2}$
BC	-1	1	-1
BD	-2	$\sqrt{2}$	$-\sqrt{2}$
CD	1	1	1
CE	2	$\sqrt{2}$	$\sqrt{2}$
DE	-1	1	-1

Resolving forces at E :

$\sum F_x : \; -\sqrt{2}\sin 45 = -1$

$\sum F_y : \; 1 - \sqrt{2}\cos 45 = 0$

$\therefore$ external force at E = 1, horizontally to the right

4(a) TUBE : $I_{xx} = I_{yy} = \frac{\pi(R_o^4 - R_i^4)}{4} = \frac{\pi(0.105^4 - 0.1^4)}{4}$

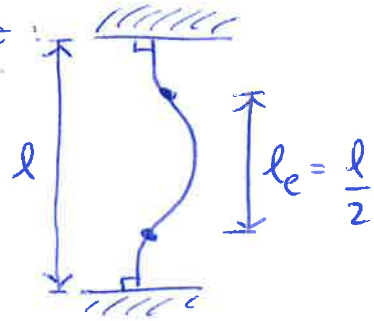
$= 1.693 \times 10^{-5} m^4$

RECTANGULAR BAR : $I_{yy} = \frac{bd^3}{12} = \frac{0.2 \times 0.08^3}{12}$

$= 8.533 \times 10^{-6} m^4$

(b) FROM (a) BAR BUCKLES BEFORE TUBE

NOTE :



$P_{cr} = \frac{\pi^2 EI}{l_e^2}$

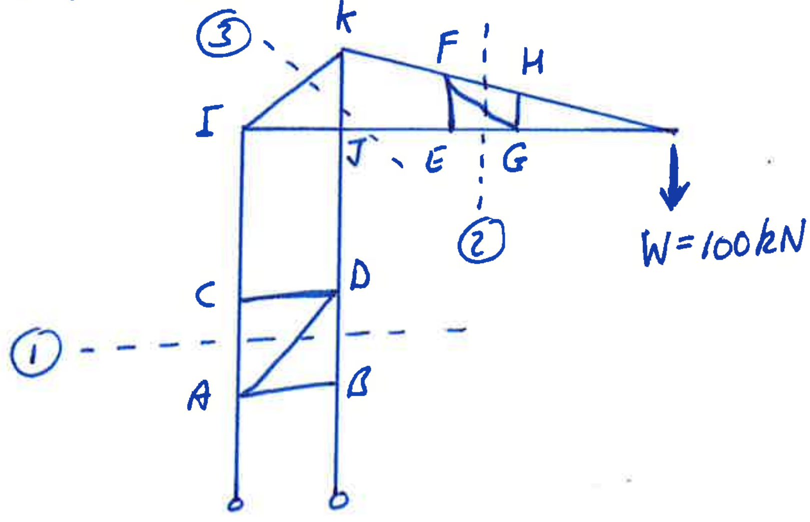
$= \frac{\pi^2 \times 210 \times 10^9 \times 8.533 \times 10^{-6}}{(1)^2}$

$= 17.69 \times 10^6 N$

$\varepsilon = \frac{\sigma}{E} \Rightarrow \delta = \frac{Fl}{AE} = \frac{17.69 \times 10^6 \times 2}{(0.2 \times 0.08) \times 210 \times 10^9}$

$= 10.53 \times 10^{-3} m$

5. (a) Define three sections through the structure:



(i) cut at section ①, and consider the structure above:

By horizontal eqn: $T_{AD} = 0$

$\therefore$ Resolving horizontally at A, $T_{AB} = 0$

Moments about B: $T_{CA} = 6W = 600\,kN$

moments about A: $T_{BD} + 7W = 0 \therefore T_{BD} = -700\,kN$

(ii) cut at section ② and consider structure to the right:

Moments about X $\Rightarrow$ $T_{FG} = 0$

$\therefore$ resolving ⊥ to FH at F, $T_{EF} = 0$

Vertical eqn, $T_{FH} \cdot \frac{1}{\sqrt{37}} = W \quad \therefore T_{FH} = 608\,kN$

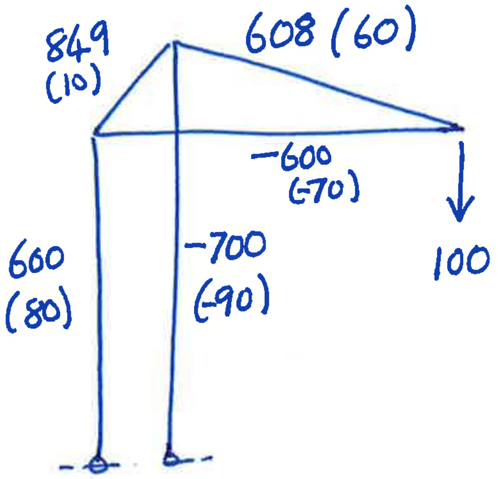
~~Horizonta~~ Moments about K, $T_{EG} + 6W = 0 \therefore T_{EG} = -600\,kN$

(iii) Resolving vertically and horizontally at J,

$T_{KJ} = T_{BD} = -700\,kN, \quad T_{IJ} = T_{EG} = -700\,kN$

cut at ③ and moments about J, $\frac{T_{IK}}{\sqrt{2}} = 6W, \therefore T_{IK} = 849\,kN$

(b) Summarising the loaded part of the structure from (a), with extensions in brackets:



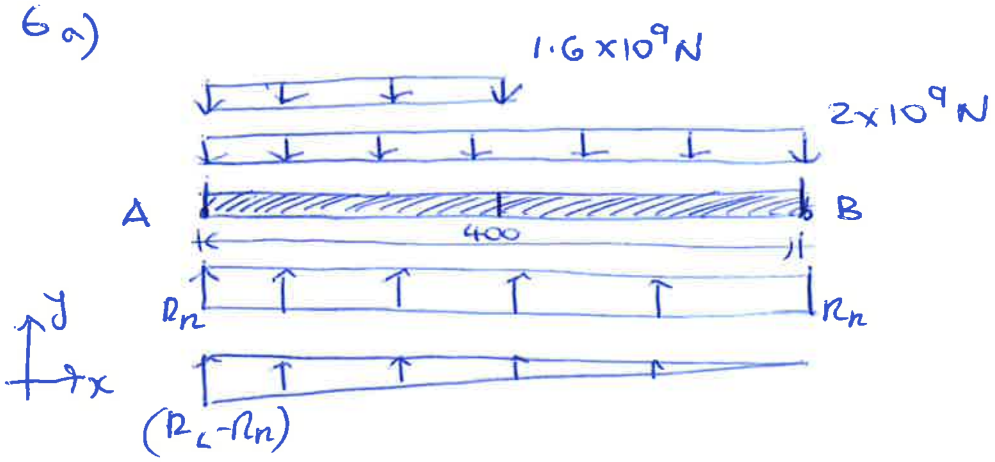
External work done by load = Internal work in bars:

$$F.\delta = \sum T.e$$

$$\therefore 100.\delta_{load} = 600 \times 80 + 700 \times 90 + 849 \times 10 + 600 \times 70 + 608 \times 80$$

$$\therefore \underline{\underline{\delta_{load} = 1{\cdot}98\text{m}}}$$

6 a)



$$\sum F_y = 0 : \quad 1.6\times10^9 + 2\times10^9 = \frac{400}{2}(R_R + R_L)$$

$$\therefore R_R + R_L = 1.8\times10^7 \text{ N/m} \quad -① $$

$\sum M = 0$ @ EITHER END OF TANKER

$$\therefore \sum M_A = 0 : (1.6\times10^9)100 + (2\times10^9)200 = 400 R_R \times 200 + \frac{400}{2}(R_L - R_R)\times\frac{400}{3}$$

$$\therefore 5.6\times10^{11} = 53{,}333 R_R + 26{,}667 R_L \quad -②$$

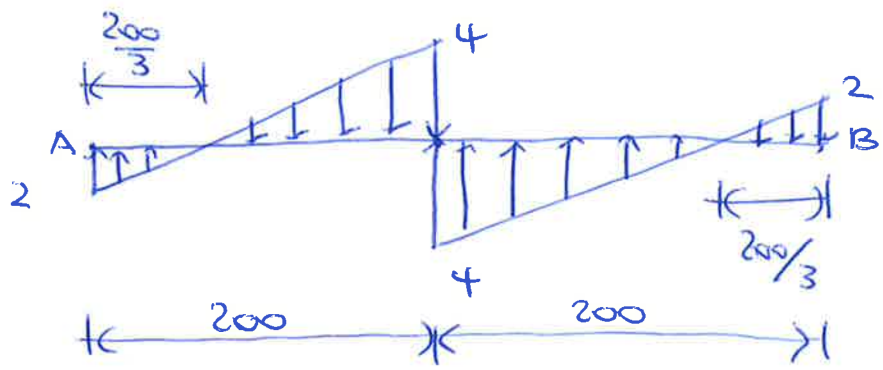
SUB ① INTO ②:

$$8\times10^{10} = 26{,}667 R_R$$

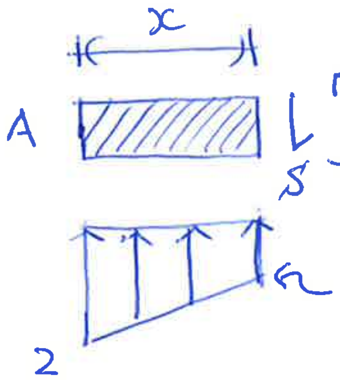
$$\therefore R_R = 3\times10^6 \text{ N/m} \quad (3 \text{ MN/m})$$

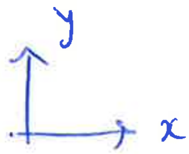
$$\& \quad R_L = 15\times10^6 \text{ N/m} \quad (15 \text{ MN/m}).$$

(b) NET LOADS & REACTIONS ON SHIP:



(ALL IN MN/m)





FROM SIMILAR TRIANGLES $= \dfrac{\left(\frac{200}{3} - x\right)}{\left(\frac{200}{3}\right)} \cdot 2$

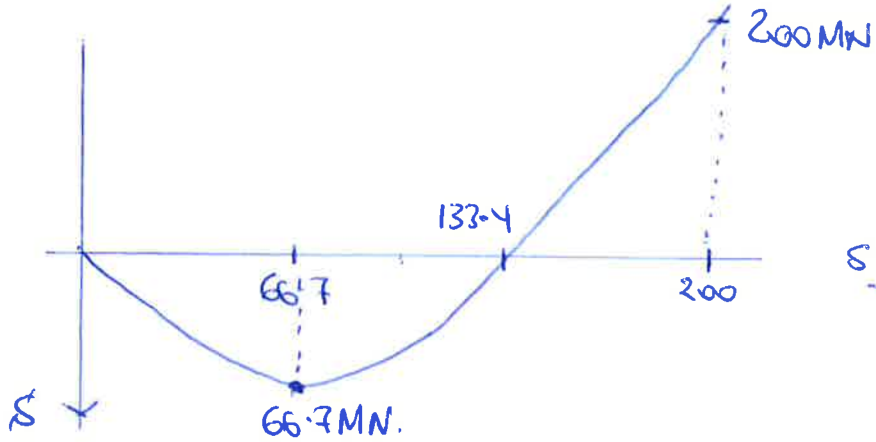
$= \left(1 - \dfrac{3x}{200}\right)2$

$\sum F_y = 0 \; : \; S = \left[\dfrac{2 + 2\left(1 - \frac{3x}{200}\right)}{2}\right]x$

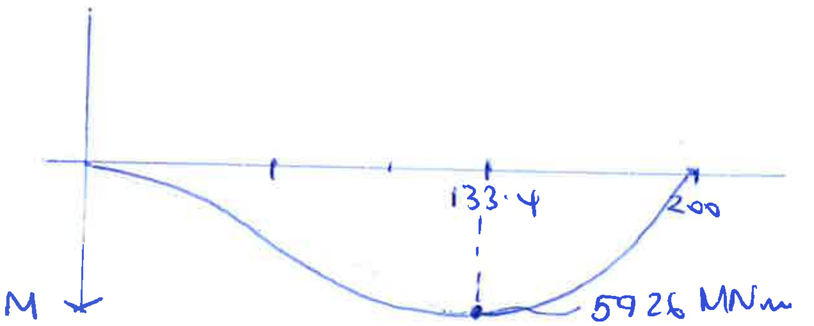
$= \left(2x - \dfrac{3x^2}{200}\right)$ MN or $\left(2x - 0.015x^2\right)$ MN

$M = \int S\,dx = x^2 - \dfrac{x^3}{200} + C \rightarrow$ AT $x = 0$, $M = 0$ $\therefore C = 0$

$= \left(x^2 - \dfrac{x^3}{200}\right)$ MNm or $\left(x^2 - 0.005x^3\right)$ MNm

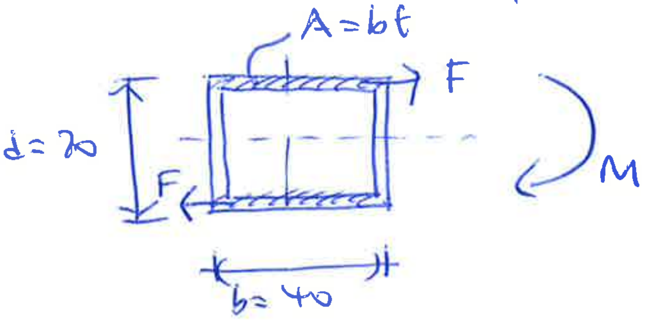


SFD



BMD

(c) BENDING CAPACITY CHECK (TOP & BOTTOM PLATES)



$$F.d \geq M$$

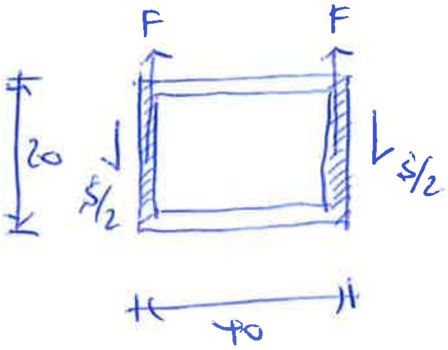
$$\sigma_y tb.d \geq M$$

$$\therefore t \geq \frac{M}{\sigma_y b d}$$

$$= \frac{5926 \times 10^6 \, Nm}{350 \times 10^6 \, N/m^2 \times 40m \times 20m}$$

$$= 21.2 \, mm.$$

SHEAR CAPACITY CHECK (SIDE PLATES)



$$F \geq S$$

$$\sigma_y . 2t . d \geq S$$

NOTE SHEAR STRENGTH OF STEEL $\simeq \frac{\sigma_y}{\sqrt{3}}$ BUT THIS IS ONLY COVERED IN IIA ∴ TAKE AS σ_y.

WITH $\sigma_y/\sqrt{3}$ $t = 12.37mm$

$$\therefore t \geq \frac{S}{\sigma_y . 2 . d}$$

$$= \frac{200 \times 10^6 N}{350 \times 10^6 \times 2 \times 40}$$

$$= 7.14 \, mm$$

∴ $t = 21.2$ mm THROUGHOUT.

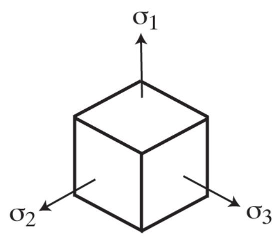
ASSUMPTIONS:
- PLANE SECTIONS REMAIN PLANE (NO SHEAR DEFORMATIONS)
- ASSUME CROSS-SECTION IS HOLLOW RECTANGLE i.e. IGNORE CURVED CORNERS AND PARAPETS.
- IGNORE PLATE BUCKLING.
- IGNORE LOCAL BENDING STIFFNESS (i.e. $t << d$; $t << b$)

7 (**short**)

(a) Strains due to stress σ_1 : $\varepsilon_1 = \frac{\sigma_1}{E},\ \varepsilon_2 = -\frac{\nu\sigma_1}{E},\ \varepsilon_3 = -\frac{\nu\sigma_1}{E}$

Strains due to stress σ_2 : $\varepsilon_1 = -\frac{\nu\sigma_2}{E},\ \varepsilon_2 = \frac{\sigma_2}{E},\ \varepsilon_3 = -\frac{\nu\sigma_2}{E}$

Strains due to stress σ_3 : $\varepsilon_1 = -\frac{\nu\sigma_3}{E},\ \varepsilon_2 = -\frac{\nu\sigma_3}{E},\ \varepsilon_3 = \frac{\sigma_3}{E}$



By superposition of stresses, the combined strains are

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

$$\varepsilon_2 = \frac{1}{E}(-\nu\sigma_1 + \sigma_2 - \nu\sigma_3)$$

$$\varepsilon_3 = \frac{1}{E}(-\nu\sigma_1 - \nu\sigma_2 + \sigma_3)$$

(b)

$$\varepsilon_1 \neq 0,\ \varepsilon_2 = \varepsilon_3 = 0$$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) \quad (1)$$

$$\varepsilon_2 = 0 = \frac{1}{E}(-\nu\sigma_1 + \sigma_2 - \nu\sigma_3) \Rightarrow \sigma_2 = \nu(\sigma_1 + \sigma_3) \quad (2)$$

$$\varepsilon_3 = 0 = \frac{1}{E}(-\nu\sigma_1 - \nu\sigma_2 + \sigma_3) \Rightarrow \sigma_3 = \nu(\sigma_1 + \sigma_2) \quad (3)$$

(2)+(3)

$$\sigma_2 + \sigma_3 = \nu(\sigma_1 + \sigma_3 + \sigma_1 + \sigma_2) = \nu(2\sigma_1 + \sigma_3 + \sigma_2)$$

$$\therefore \sigma_2 + \sigma_3 = \frac{2\nu\sigma_1}{1-\nu}$$

Substitute into (1)

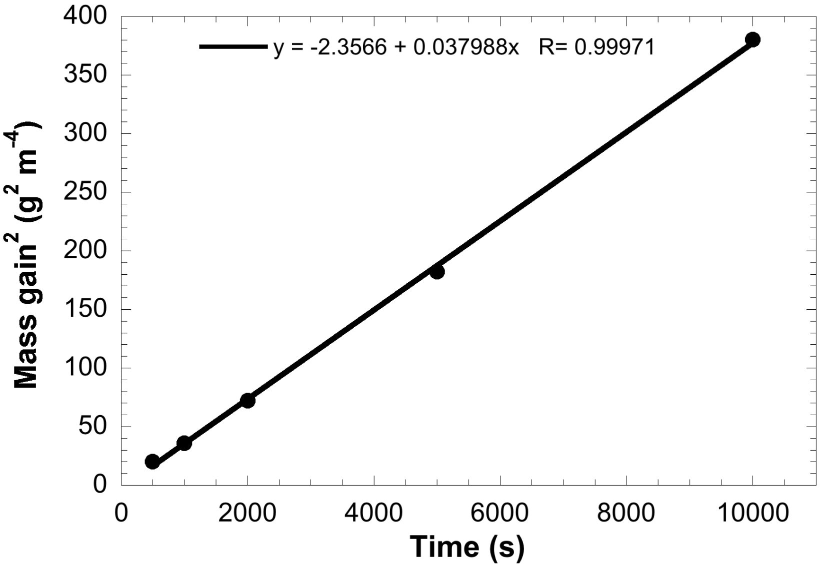
$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\left[\frac{2\nu\sigma_1}{1-\nu}\right]\right) = \frac{1}{E}\left(\sigma_1\left[1 - \frac{2\nu^2}{1-\nu}\right]\right) = \frac{1}{E}\left(\sigma_1\left[\frac{1-\nu-2\nu^2}{1-\nu}\right]\right)$$

$$= \frac{\sigma_1}{E}\left(\frac{(1+\nu)(1-2\nu)}{1-\nu}\right)$$

$$\therefore \frac{\sigma_1}{\varepsilon_1} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

Comments: This question was answered reasonably well, although part (a) was often a copy-paste of the answer without any explanation. We were expecting the candidates to clearly separate the contributions of the stresses in each direction. Almost all the candidates recognised that the constrains in the transverse direction led to zero strain in part (b), but the subsequent algebra used to derive the simple formula generally lacked efficiency, leading to a loss of precious minutes.

8 (**short**)
(a)



From the gradient, the parabolic rate constant is ~ 0.038 g^2 m^{-4} s^{-1}.
(b)

$$\Delta m^2 = kt = 0.038 \times (300 \times 3600 \times 24) \text{ g}^2 \text{ m}^{-4}$$

$$\therefore \Delta m = 0.992 \text{ (kg of O) m}^{-2}$$

Now

$$\Delta m = 0.992 \text{ (kg of O) m}^{-2} =$$

$$= \frac{0.992 \times N_A}{16} \text{ (kmole of O) m}^{-2}$$

$$= \frac{0.992 \times N_A}{16 \times 4} \text{ (kmole of } Fe_3O_4\text{) m}^{-2}$$

$$= \frac{0.992 \times N_A}{16 \times 4} \cdot \frac{3 \times 56}{N_A} \text{(kg of Fe) m}^{-2}$$

$$= \frac{0.992 \times 3 \times 56}{16 \times 4 \times 7800} \text{ (m of Fe)}$$

=0.33 mm (on either side) of Fe lost < 1 mm

Comments: A large number of candidates were penalised on part (a): drawing what "looks like" a square root function is obviously not enough. Plotting the squared mass versus time gives a line, which is much less ambiguous, and easier to draw! Also calculating the average of k without a *relative measure of the deviation in the data is inconclusive: the sample standard deviation, for example, is covered in paper 4.*

9 (**short**)

(a) Consider D_1 the diameter of A and D_2 the diameter of B. The length a of the edge of the AB unit cell should be $a = D_1 + D_2$. The diagonal of one of the faces of the unit cell should be $\sqrt{2}a = 2D_1$. Hence

$$\sqrt{2}a = 2D_1 \Rightarrow a = \sqrt{2}D_1$$

$$D_1 + D_2 = \sqrt{2}D_1$$

$$D_2 = 0.414 D_1$$

$$\therefore \frac{D_2}{D_1} = 0.414$$

(b) Number of silicon atoms per unit cell: $8\times 1/8$ (corners) $+6\times 1/2$ (faces) = 4
Therefore, there are 4 silicon atoms and 4 carbons in the unit cell.

$$\textit{theoretical density} = \frac{\text{mass of the unit cell}}{\text{volume of the unit cell}}$$

$$\text{mass of the unit cell}= = \frac{\left[(4\times 28.09 + (4\times 12.01)\right]\cdot 10^{-3}}{6.02\times 10^{23}} \text{ kg}$$

$$\text{volume of the unit cell}=(0.436\times 10^{-9})^3 \text{ m}^3$$

Hence

$$\textit{theoretical density} = \frac{\left[(4\times 28.09 + (4\times 12.01)\right]\cdot 10^{-3}}{(0.436\times 10^{-9})^3 \cdot 6.02\times 10^{23}}$$

$$\approx 3.21 \text{ Mg m}^{-3}$$

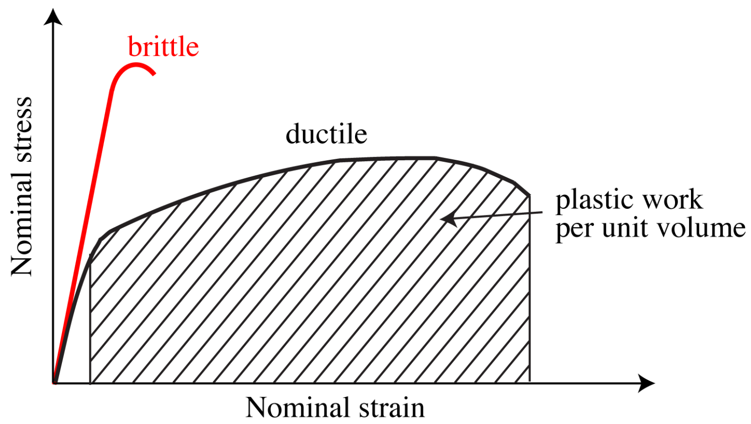
Comments: This was an easy question, usually well answered. We were severely penalising unrealistic results: diameter of B much bigger that diameter of A, mass density of a few grams per cubic meter… We have seen numerous calculations ran very inefficiently here as well.

10 (**short**)

(a) Ductile Fracture: The fracture surface of Fig. 8(A) shows evidence of extensive plastic flow. The cell-like features are regions in which voids formed (probably nucleated by decohesion at inclusion interfaces, some of which can be seen in the micrograph), with final failure occurring when the voids grow and coalesce. This facilitates extensive plasticity ahead of the crack tip prior to fracture, with large amounts of energy being absorbed in the process – see schematic below (significant plastic work per unit volume). This metal will exhibit the largest fracture energy.

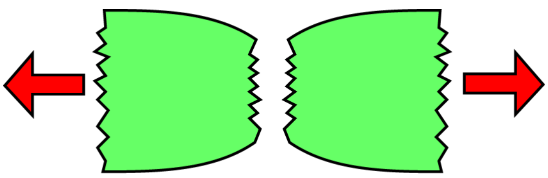
Brittle Fracture: The flat faceted fracture surface of Fig. 8(A) shows little evidence of plastic flow. The fracture energy is therefore likely to be relatively low – certainly lower than that of Fig. 8(A).

Nominal stress-strain curves would have the forms shown below:

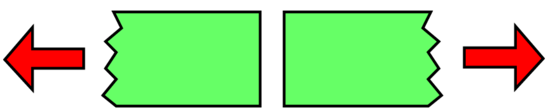


(b) Temperature is responsible for the differences between the two micrographs. Presumably in Fig. 8(A), the sample was tested at or above room temperature whereas in Fig. 8(B) the sample was tested close or below the ductile-brittle transition temperature. It is well known that steel becomes brittle below the ductile-brittle transition temperature, as the ease of dislocation motion is sufficiently reduced.

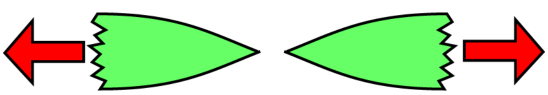
Macroscopic fracture surfaces: Failure of engineering metals does commonly involve fracture, often after extensive plastic flow and some necking have occurred – see schematic below.



In brittle failure, the fracture surface is planar on a macro scale- see schematic below.



(NB: Very ductile materials, if ultra pure, often fail by ductile rupture (progressive necking down to a point), with little or no crack propagation as such. However, such highly ductile materials are too soft to be useful for most purposes.)



Comments: We were expecting more from the candidates since they had a lab covering this question. This was clearly an opportunity to show their knowledge, which ended up showing some significant gaps.

- Part (a): a clear description of the micrographs, taking advantage of the scale bar. In A, visible micron size voids nucleated by impurities/inclusions. Plasticity due to plane sliding past each other thanks to the growing number of moving dislocations (which are not visible on the micrograph, contrary to a popular belief). Explain how the stress vs. strain curve can be used to evaluate the energy expended during fracture. In B, a flat faceted surface is shown with little evidence of plastic flow.

- Part (b): temperature-dependent ductile-brittle transition in bcc steel, as opposed to fcc metals such as copper. Ductile fracture happens after necking.

11 (**long**)

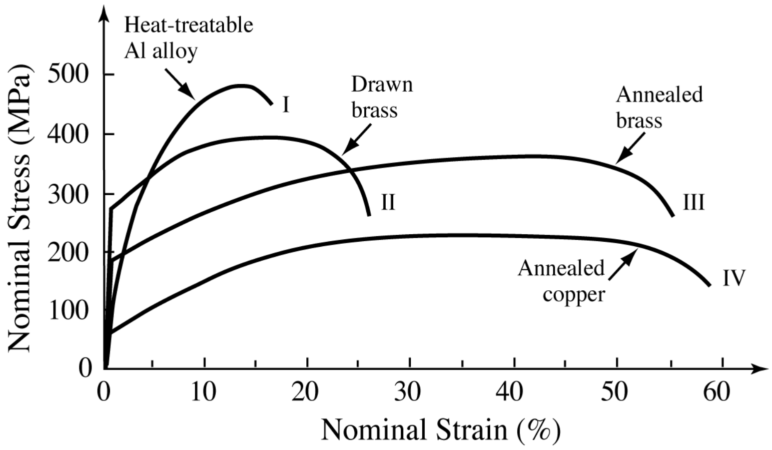
(a)

I – Heat-treatable Al alloy

II – Brass, drawn

III – Brass. annealed

IV – Copper, annealed



A heat–treatable Al alloy would have a high yield strength due to precipitation hardening. Drawn brass will have a high dislocation density due to work hardening. The yield stress is therefore higher than that of annealed brass and significantly higher than that annealed copper. Annealing causes a sharp reduction in the dislocation density, reducing the yield stress (by making dislocation motion easier) and also raising the failure strain (ductility). Annealed brass would have a higher yield stress from annealed copper, due to solid solution strengthening from the Zn.

(b)

(i)

$$\frac{d\sigma_t}{d\varepsilon_t} = \sigma_t \Rightarrow 125\ \varepsilon_t^{-1/2} = 250\ \varepsilon_t^{1/2}$$

$$\therefore \varepsilon_t = \frac{1}{2} \text{ at maximum load}$$

(ii)

$$\sigma_{\mathrm{n}} = \frac{F}{A_{\mathrm{o}}} \text{ and } \sigma_{\mathrm{t}} = \frac{F}{A}$$

$$AL = A_{\mathrm{o}} L_{\mathrm{o}}$$

$$\sigma_{\mathrm{t}} = \sigma_{\mathrm{n}} \left(\frac{A}{A_{\mathrm{o}}} \right) = \sigma_{\mathrm{n}} \left(\frac{L_{\mathrm{o}}}{L} \right)$$

Nominal strain

$$\varepsilon_{\mathrm{n}} = \frac{L - L_{\mathrm{o}}}{L_{\mathrm{o}}} = \frac{L}{L_{\mathrm{o}}} - 1 \Rightarrow \frac{L}{L_{\mathrm{o}}} = 1 + \varepsilon_{\mathrm{n}}$$

$$\varepsilon_{\mathrm{t}} = \int_{L_{\mathrm{o}}}^{L} \frac{dL}{L} = \ln\left(\frac{L}{L_{\mathrm{o}}} \right)$$

$$\varepsilon_{\mathrm{t}} = \ln\left(1 + \varepsilon_{\mathrm{n}}\right) \ \& \ \sigma_{\mathrm{t}} = \sigma_{\mathrm{n}} \left(1 + \varepsilon_{\mathrm{n}}\right)$$

$$\sigma_{\mathrm{n}} = \frac{\sigma_{\mathrm{t}}}{1 + \varepsilon_{\mathrm{n}}} = \frac{250 \ \varepsilon_{\mathrm{t}}^{1/2}}{\exp \varepsilon_{\mathrm{t}}} = \frac{250 \ 0.5^{1/2}}{\exp 0.5} \approx 107 \text{ MPa}$$

(c) $\Delta\sigma \ N_f^{\alpha} = C_1$

1st set of tests:

$$0.5\sigma_{\mathrm{ts}}(10^6)^{\mathrm{m}} = 0.65\sigma_{\mathrm{ts}}(10^4)^{\mathrm{m}} = C_1$$

$$\Rightarrow 10^{2\mathrm{m}} = 1.3$$

$$2\mathrm{m} \log 10 = \log 1.3$$

$$\therefore \mathrm{m} = \frac{\log 1.3}{2 \log 10} = 0.057$$

$$0.5\sigma_{\mathrm{ts}}(10^6)^{0.057} = C_1$$

$$\Rightarrow C_1 = 1.1\sigma_{\mathrm{ts}}$$

2nd set of tests:

Use Goodman's rule to obtain zero mean stress conditions

$$\Delta\sigma = \Delta\sigma_{\mathrm{o}} \left(1 - \frac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{ts}}} \right) \Rightarrow \Delta\sigma_{\mathrm{o}} = \frac{\Delta\sigma}{\left(1 - \frac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{ts}}} \right)}$$

$$\sigma_{\mathrm{m}} = 0.1\sigma_{\mathrm{ts}} \text{ and } \Delta\sigma = 0.5\sigma_{\mathrm{ts}}$$

$$\therefore \Delta\sigma_{\mathrm{o}} = \frac{0.5}{0.9} \sigma_{\mathrm{ts}}$$

Using Basquin's law

$$\frac{0.5}{0.9} \sigma_{\mathrm{ts}} \, N_f^{0.057} = 1.1\sigma_{\mathrm{ts}} \Rightarrow N_f^{0.057} = 1.98$$

$$\therefore N_f = 1.6 \times 10^5 \text{ cycles}$$

$\sigma_m = 0.1\sigma_{ts}$ and $\Delta\sigma = 0.55\sigma_{ts}$

$$\therefore \Delta\sigma_o = \frac{0.55}{0.9}\sigma_{ts}$$

$$\frac{0.55}{0.9}\sigma_{ts} N_f^{0.057} = 1.1\sigma_{ts} \Rightarrow N_f^{0.057} = 1.8$$

$$\therefore N_f = 3\times10^4 \text{ cycles}$$

Miner's Rule:

$$\sum_i \frac{N_i}{N_{fi}} = 1 \Rightarrow \frac{4\times10^4}{1.6\times10^5} + \frac{N}{3.0\times10^4} = 1$$

$$\therefore N = 2.25\times10^4 \text{ cycles}$$

Hence total number of cycles to failure is $4\times10^4 + 2.25\times10^4 = 6.25\times10^4$ cycles

Comments: Part (a) was done reasonably well. Most candidates were able to identify which curve corresponds to which alloy, but very few candidates could explain correctly the differences in the curves. A common mistake was to relate hardening effects to stiffness, while hardness is only related to the yield strength of the material. Surprisingly, very few students identified "precipitation hardening" as the hardening mechanism for the heat-treatable Al alloy, while several candidates referred to "work hardening" instead. For the drawn brass, several candidates talked about hardening effects due to alignment of molecules rather than the increase in dislocation density due to work hardening. Parts b(i) and b(ii) were not answered well, particularly Part b(ii), and were the major source of lost marks. A lot of candidates integrated the relationship between the true stress and true strain instead of differentiating. Some candidates calculated just the true stress using the true stain from b(i) and didn't estimate the tensile strength (based on the nominal stress). Part (c) was done reasonably well, with many candidates scoring highly. A large number of candidates did not use or didn't use correctly Goodman's rule to calculate the stress range at a zero mean stress for the assessment of the fatigue life in the second series of tests. Quite a few candidates made numerical errors in the calculations, which led to incorrect values for Basquin's Law constants and subsequent fatigue life.

12 (**long**)

(a) The main concern with any pressurised vessel is to avoid catastrophic rupture, particularly when the pressurised fluid is a gas - in which case an explosion is likely to result. Such rupture most commonly occurs in the form of fast crack propagation. One guideline which is commonly used in designing high-pressure systems is the "leak-before-break" criterion. The idea involved here is that a crack should not be able to propagate under fast fracture conditions before it has grown sufficiently to penetrate through the complete wall thickness, when leakage will occur, causing a drop in pressure. This will reduce the stress levels so the system should be failsafe and an explosion impossible.

So if the critical flaw size for fast fracture a_{crit} is less than the wall thickness *t* of the vessel, then fast fracture can occur with no warning. But if the critical size is greater than *t* then gas will leak out through the crack before the crack is large enough to propagate under fast fracture conditions.

(b) Tensile stress σ (P = 3 MPa, r = 1 m, t = 10 mm)

$$\sigma = \frac{Pr}{2t} = \frac{3 \cdot 1000}{2 \cdot 10} = 150 \text{ MPa}$$

Note: For a thin-walled spherical vessel ($t << r$), the relationship between internal pressure P and the stresses in the wall σ are readily derived by balancing the forces exerted by the pressure ($\pi r^2 P$) and by the stresses in the wall ($\sigma 2\pi rt$).

$$2\pi rt\sigma = \pi r^2 P \Rightarrow \sigma = \frac{Pr}{2t}$$

(i) For leak $a = t = 10$ mm

$$K = 1.13 \cdot 150 \cdot \sqrt{\pi \cdot (10 \times 10^{-3})} \approx 30 \text{ MPa}\sqrt{\text{m}}$$

Since $K < K_{IC} = 85 \text{ MPa}\sqrt{\text{m}}$, the crack will satisfy the "leak before break" criterion.

Alternatively, a_{crit} can be estimated:

$$a_{crit} = \left(\frac{K_{IC}}{1.13 \cdot 150}\right)^2 \cdot \frac{1}{\pi} = \left(\frac{85}{1.13 \cdot 150}\right)^2 \cdot \frac{1}{\pi} \approx 0.08 \text{ m}$$

Since $a_{crit} = 80 \text{ mm} > t = 10 \text{ mm}$, the crack will satisfy the "leak before break" criterion.

(ii) $\frac{da}{dN} = A\Delta K^4$ where $A = 5 \times 10^{-16} (\text{m/cycle}) \left(\text{MPa}\sqrt{\text{m}}\right)^{-4}$

$$\frac{da}{dN} = 5 \times 10^{-16} \cdot (1.13^4 \cdot 150^4 \cdot \pi^2 a^2)$$

$$\int_{2\times10^{-3}}^{10\times10^{-3}} \frac{da}{a^2} = (5 \times 10^{-16} \cdot 1.13^4 \cdot 150^4 \cdot \pi^2) \int_0^{N_f} dN$$

$$N_f = \frac{1}{5 \times 10^{-16} \cdot 1.13^4 \cdot 150^4 \cdot \pi^2} \left[-\frac{1}{a}\right]_{2\times10^{-3}}^{10^{-2}} = \frac{1}{4.07332 \times 10^{-6}} (500 - 100)$$

$$\therefore N_f \approx 9.81 \times 10^7 \text{ cycles}$$

(c) If the crack length, a , is set equal to the wall thickness, t , then the "leak before break criterion" can be written

$$K = Y\sigma\sqrt{\pi t} \leq K_{IC} \qquad (33)$$

Using the expression for the stress in the wall $\left(\sigma = \frac{Pr}{2t}\right)$ to substitute for t gives

$$Y\sigma\sqrt{\pi\left(\frac{Pr}{2\sigma}\right)} = Y\sqrt{\frac{\sigma\pi Pr}{2}} \leq K_{IC}$$

If it is also required that plastic deformation should not occur, so that this stress level can be set equal to the yield stress, σ_y , then an expression can be obtained for the *maximum operating pressure*

$$P_{max} = \frac{2K_{IC}^2}{Y^2 \sigma_y \pi r} = \frac{2}{Y^2 \pi r}\left(\frac{K_{IC}^2}{\sigma_y}\right) \tag{35}$$

The ratio K_{IC}^2/σ_y thus represents a material merit index for being able to operate safely under high pressure. As shown in the Table below, the medium carbon steel will contain the greatest pressures.

Material	σ_y (MPa)	K_{IC} (MPa$\sqrt{m}$)	$\frac{K_{IC}^2}{\sigma_y}$ (MPa m)
Medium carbon steel	525	75	9.3
Titanium alloy	740	65	5.7
Aluminium alloy	265	30	3.4
Magnesium Alloy	235	20	1.7

Comments: Part (a) was generally done very well while parts (b)(i) and b(ii) were answered reasonably well. The main shortcoming of this question was part (c). In parts b(i) and b(ii), some candidates used the pressure value in their calculations and several of them made numerical errors in their calculations. In Part b(ii), some candidates used the critical flaw size for fast fracture (from b(i)) as the upper limit for the integral instead of the vessel wall thickness. In part (c), most candidates were running out of time when they reached this question. A large number of candidates derived the correct material merit index (as this was covered in the lectures) but very few of them did this properly using the leak-before-break criterion (setting the crack length, a, equal to the wall thickness, t and stating that $K = Y\sigma\sqrt{\pi t} \le K_{IC}$*). Marks were lost because of lack of detail.*