ENGINEERING TRIPOS PART IA 2014

Paper 4 Mathematical Methods

Solutions

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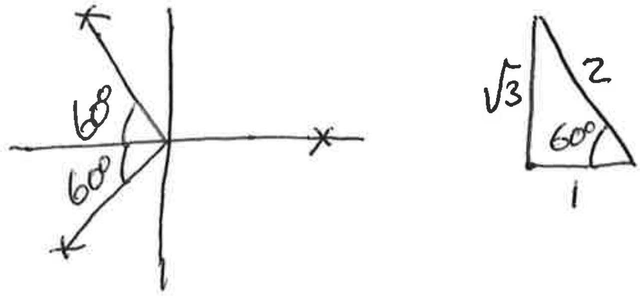
1/ $a^3 = 1$

$1 = e^{j(0+2n\pi)}, \quad n = 0, 1, 2$

$a = e^{j\frac{2}{3}n\pi}$



$$\text{Det } A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{vmatrix} = 1(a^4 - a^2) - 1(a^2 - a) + 1(a - a^2)$$

$$= a^4 - 3a^2 + 2a$$

$$= 3(a - a^2)$$

Note $a^2 = (e^{j2\pi/3})^2 = e^{j4\pi/3}$

$= (e^{j4\pi/3})^2 = e^{j8\pi/3} = e^{j2\pi/3}$

$a = 1$; Det $A = 0$

$a = e^{j2\pi/3}$; Det $A = 3(e^{j2\pi/3} - e^{j4\pi/3})$

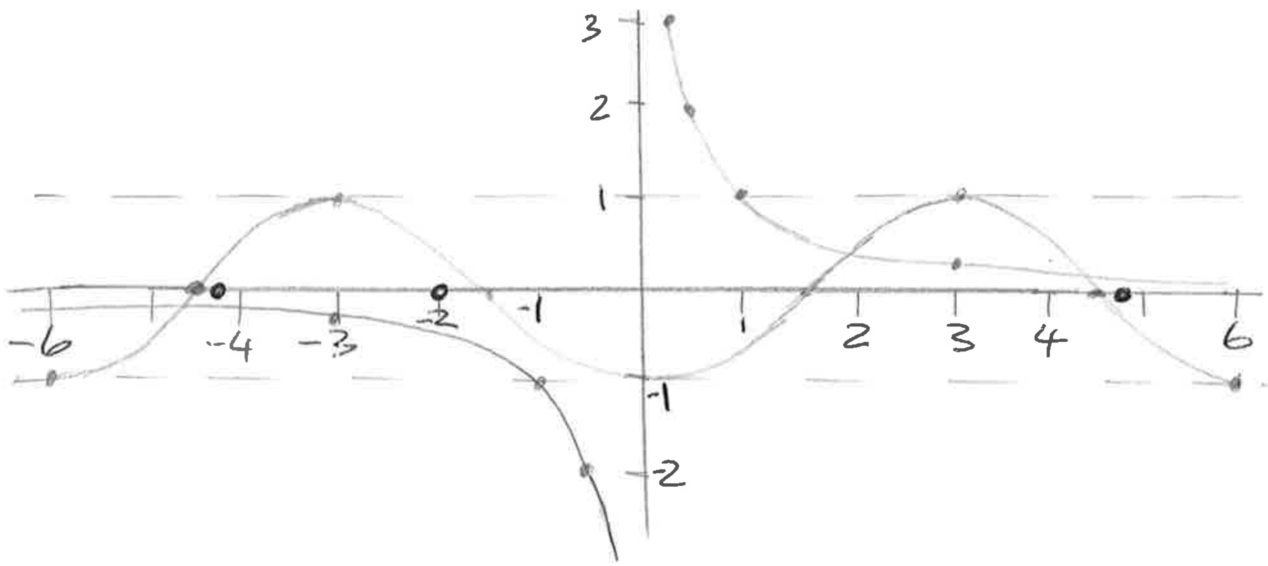
$= 3\left(2 \times \frac{\sqrt{3}}{2}\right)j = 3\sqrt{3}j$

$a = e^{j4\pi/3}$; Det $A = 3(e^{j4\pi/3} - e^{j2\pi/3})$

$= -3\sqrt{3}j$

2/ $y = \frac{1}{x} - \cos x$

Sketch $\frac{1}{x}$ & $\cos x$ on the same axes



Compare ~ Likely intersections at -2, -4.3

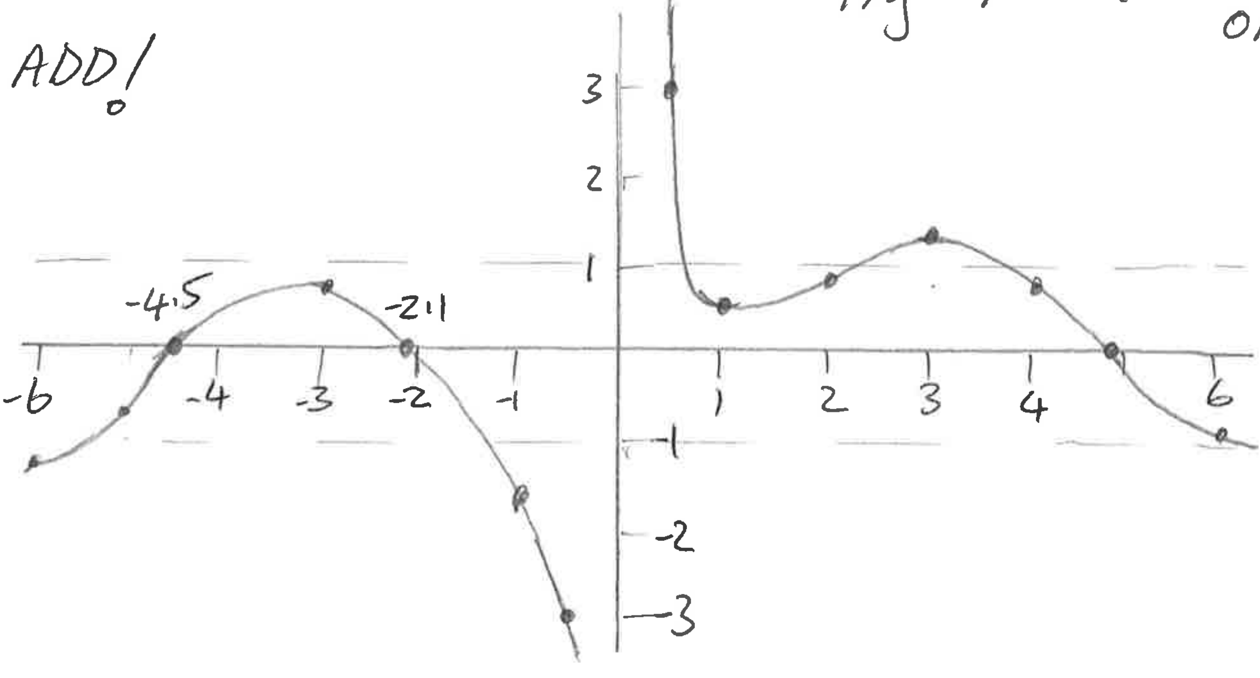
Check: -2/ $\frac{1}{-2} - \cos -2 = -0.08$ -4.3/ $\Rightarrow +0.16$

($\frac{1}{x}$ changes slowly)

$-2.1 \Rightarrow 0.03$ OK

try $-4.5 \Rightarrow 0.01$ OK.

ADD!



3/(a) $\underline{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \lambda_1 = 3$

$\underline{u}_1 \times \underline{u}_2 = \underline{u}_3$

$= \begin{vmatrix} i & j & k \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \dfrac{i}{\sqrt{2}} + \dfrac{j}{\sqrt{2}} + 0k$

$\underline{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_2 = 2$

$\lambda_3 = 1$

$A = U \Lambda U^T$

$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) As <u>Ans</u> is Real and Symmetric, it's good.

i.e. $A^T = A$ & real

$\text{Trace}[A] = 6 = \text{Det}\,\Lambda$

$A U_n = \lambda_n U_n$, for all n

$\text{Det}(A - \lambda_n I) = 0$, for all λ_n

All of the above are <u>simple</u> to perform.

4 (c) Vectors must meet at the vertex

check $\begin{pmatrix}1\\3\\0\end{pmatrix} = \underline{a} \quad \begin{pmatrix}\alpha\\0\\0\end{pmatrix} = \underline{b} \quad \begin{pmatrix}3\\1\\2\end{pmatrix} = \underline{c}$

$$\underline{b}.\,\underline{a}\times\underline{c} = \begin{vmatrix}\alpha & 0 & 0\\ 3 & 1 & 2\\ 1 & 3 & 0\end{vmatrix} = 6\alpha$$

$$Vol = \frac{1}{6}|6\alpha| = \alpha$$

(d) Area of base $= \frac{1}{2}|\underline{b}\times\underline{c}|$

$$|\underline{b}\times\underline{c}| = \begin{vmatrix} i & j & k\\ \alpha & 0 & 0\\ 3 & 1 & 2\end{vmatrix} = -j2\alpha + k\alpha$$

$$\frac{1}{2}|\underline{b}\times\underline{c}| = \frac{1}{2}\alpha\sqrt{5}$$

Height = distance of 1, 3, 0 to opposite face.

$$= \frac{3.\ Vol}{\frac{1}{2}\alpha\sqrt{5}} = \frac{6}{\sqrt{5}}$$

4 (a)

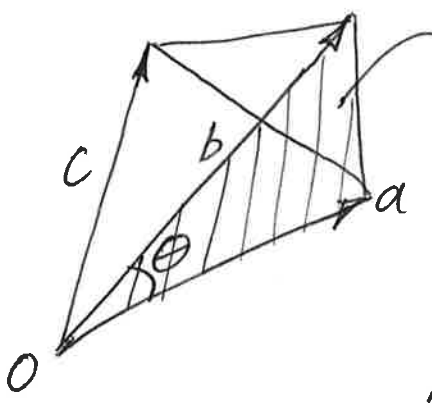
$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } z = 2y$$

solve:

$$z = \lambda$$
$$y = 3 - \lambda$$
$$z = 2y = 2(3-\lambda) = \lambda$$
$$\lambda = 2$$

$$p = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$ or by inspection.

(b)



Area $\frac{1}{2}|\underline{a}||\underline{b}|\sin\theta$

$$\frac{1}{2}|\underline{a}||\underline{b}|\sin\theta\,\hat{\underline{n}} = \frac{1}{2}(\underline{a}\times\underline{b})$$

$$\text{Area} = \left|\frac{1}{2}(\underline{a}\times\underline{b})\right|$$

Height $\underline{c}\cdot\hat{\underline{n}} = \underline{c}\cdot\dfrac{\underline{a}\times\underline{b}}{|a\times b|}$ ($\hat{n}$ above)

$$\text{Vol} = \frac{1}{3}\cdot\frac{1}{2}|(\underline{a}\times\underline{b})|\cdot\underline{c}\cdot\frac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$$

$$= \frac{1}{6}|\underline{c}\cdot\underline{a}\times\underline{b}|.$$ There are other routes.

Avoids handedness

5 (a) $\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

C.F. $V = A\cos 2t + B\sin 2t$.

P.I. Try $1/4$ – ok.

G.S. $V = A\cos 2t + B\sin 2t + 1/4$

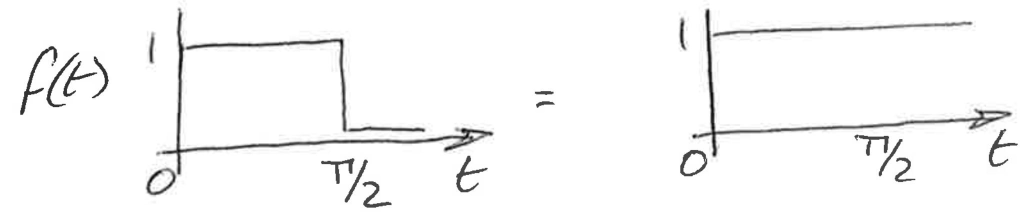
$\dot{V} = -2A\sin 2t + 2B\cos 2t$

$V(0) = 0 \quad \dot{V}(0) = 0$

$A = -1/4 \;,\quad B = 0$

$V(t) = \frac{1}{4}(1 - \cos 2t)$ for $0 < t < T$

(b) Various methods! This is reliable:



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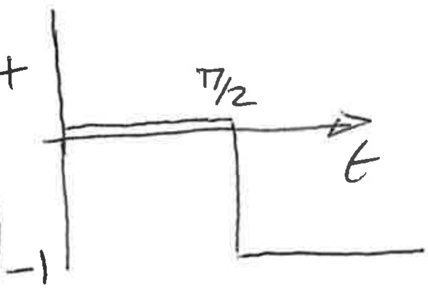


$V(t) = \frac{1}{4}(1 - \cos 2t)$

$-\frac{1}{4}\left(1 - \cos 2\left(t + \frac{\pi}{2}\right)\right)$

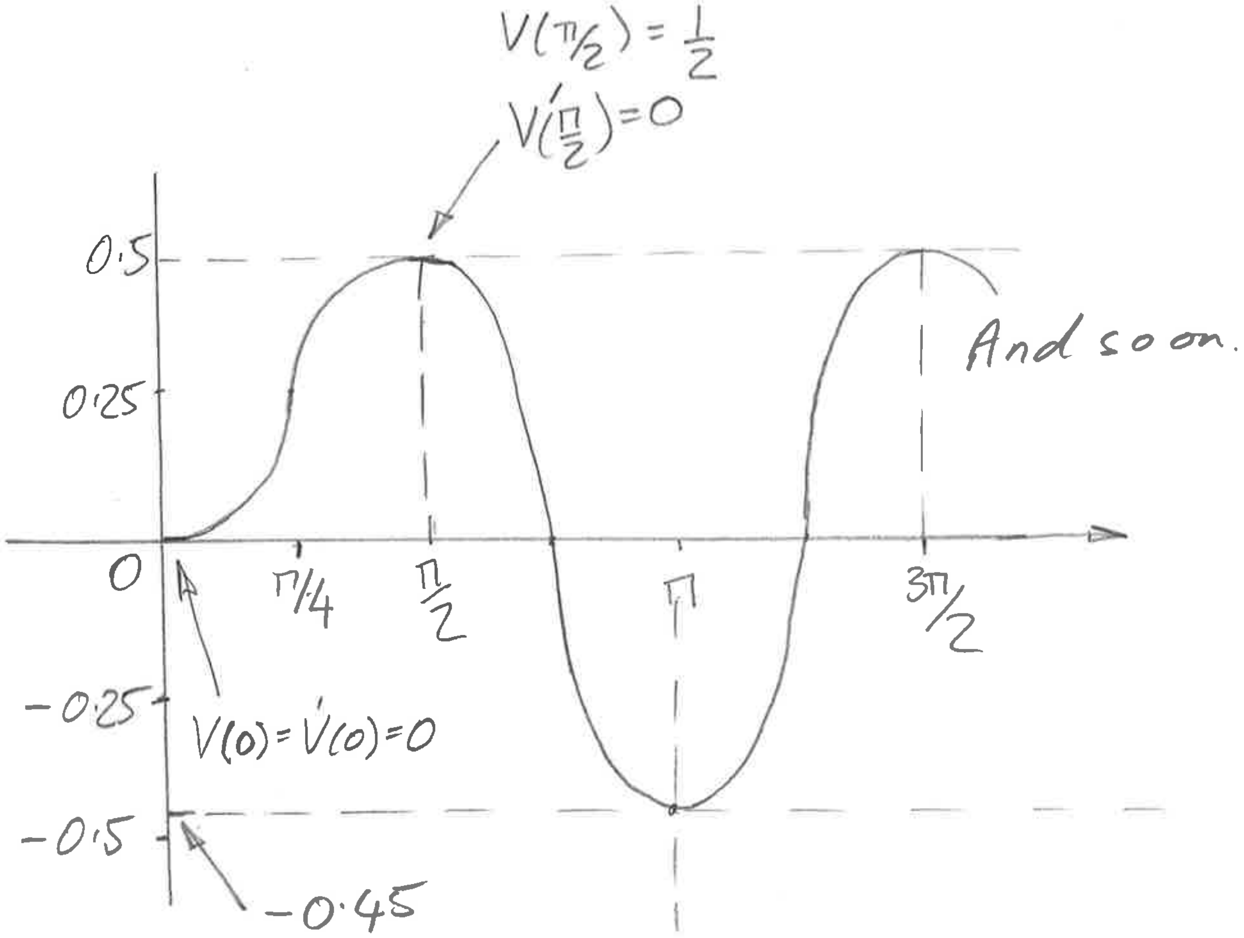
$= -\frac{1}{2}\cos 2t$

5(c) Now use same method:

$$V(t) = \frac{1}{4}(1-\cos 2t) - \frac{0.9}{4}\left(1-\cos 2\left(t-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{40} - \frac{19}{40}\cos 2t$$

Sketch! Needs care as there are details in the question.



Note method used makes sketching easier!

Section B Crib Paper 4 2014

6. Step response satisfies $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 1$

with $y(0) = \dot{y}(0) = 0$.

C.F. $e^{\lambda t} \Rightarrow \lambda^2 + 4\lambda + 4 = 0 \Rightarrow (\lambda+2)^2 = 0$

$\lambda = 2$ (twice) $\Rightarrow$ C.F. $= (At+B)e^{-2t}$

P.I. $y = \text{const} = \frac{1}{4}$

$\therefore \quad y = \frac{1}{4} + (At+B)e^{-2t}$

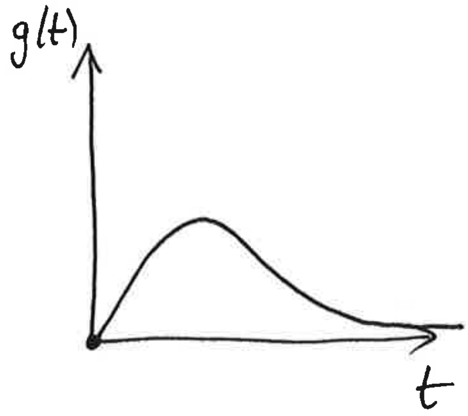
$y(0) = 0 \Rightarrow \frac{1}{4} + B = 0 \qquad \dot{y}(0) = 0 \Rightarrow -2(At+B)e^{-2t} + Ae^{-2t} = 0$ @ $t=0$

$\Rightarrow A = 2B$ i.e. $B = -\frac{1}{4}$, $A = -\frac{1}{2}$

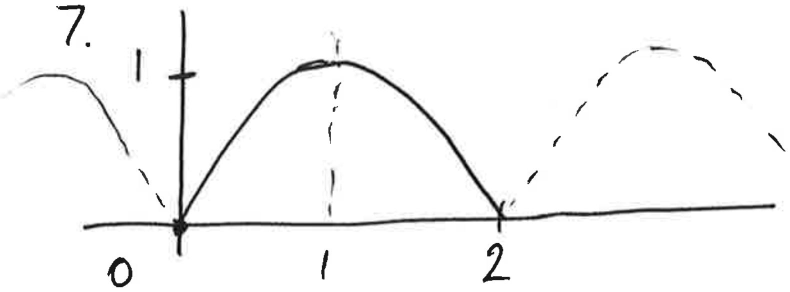
$\Rightarrow$ step response $y = \frac{1}{4} - \left(\frac{t}{2} + \frac{1}{4}\right)e^{-2t}$

Impulse response is $\frac{d}{dt}(\text{step resp}) = -\frac{1}{2}e^{-2t} + 2\left(\frac{t}{2} + \frac{1}{4}\right)e^{-2t}$

$\underline{= t\, e^{-2t}}$



7.



$$f(x) = 2x - x^2$$

(a) $f(-x) = f(2-x) = 2(2-x) - (2-x)^2$

$$= (2-x)[2-2+x] = 2x - x^2 = f(x)$$

$\Rightarrow f$ even function.

This means no odd functions (sine terms) will appear in F.S. for f.

(b) $$f(x) = d + \sum_{n=0}^{\infty} a_n \cos \frac{2\pi n x}{2} = d + \sum_{n=0}^{\infty} a_n \cos \pi n x$$

$$d = \frac{1}{2} \int_0^2 (2x - x^2)\,dx = \frac{1}{2}\left[x^2 - \frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left[4 - \frac{8}{3}\right] = \underline{\underline{\frac{2}{3}}}$$

$$a_n = \frac{2}{2} \int_0^2 (2x - x^2) \cos n\pi x \, dx$$

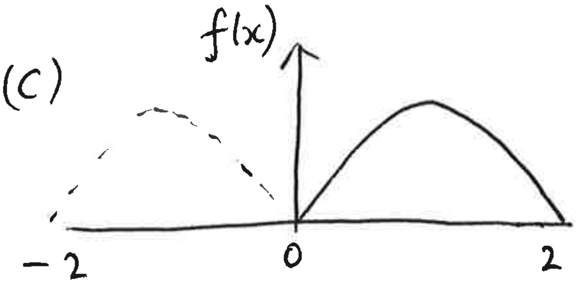
$$= \left[\frac{\sin n\pi x}{n\pi}(2x - x^2)\right]_0^2 - \int_0^2 \frac{\sin n\pi x}{n\pi}(2 - 2x)\,dx$$

$$= 0 + \left[(2-2x)\frac{\cos n\pi x}{n^2\pi^2}\right]_0^2 - \int_0^2 \frac{\cos n\pi x}{n^2\pi^2}(-2)\,dx$$

$$= \left[-\frac{2\cos 2n\pi}{n^2\pi^2} - \frac{2}{n^2\pi^2}\right] + \frac{2}{n^2\pi^2}\left[\frac{\sin n\pi x}{n\pi}\right]_0^2$$

$$= \underline{\underline{-\frac{4}{n^2\pi^2}}}$$

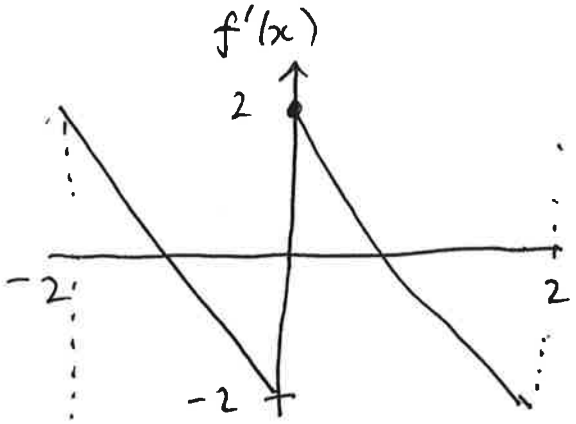
(c)



For $0 < x < 2$

$$f = 2x - x^2 \quad \text{even fn}$$

$$\Rightarrow f'(x) = 2 - 2x \quad \text{\& odd fn}$$

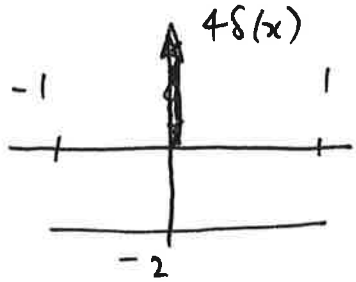


$$\Rightarrow f''(x) = -2 \quad \text{\& even function}$$

$+ 4\delta(x)$ to account for jump @ origin

+ jumps at 2, -2, etc.

$\therefore$ For $-1 < x < 1$ $\quad f''(x) = 4\delta(x) - 2$



Using $-1 < x < 1$ as range

f'' is even fn & $\int f'' dx = 0$

$$\therefore f''(x) = \sum_{n=1}^{\infty} a_n \cos n\pi x$$

where $a_n = \frac{2}{2} \int_{-1}^{1} [4\delta(x) - 2] \cos n\pi x \, dx$

$$= 4 - 2\left[\frac{\sin n\pi x}{n\pi}\right]_{-1}^{1} = 4$$

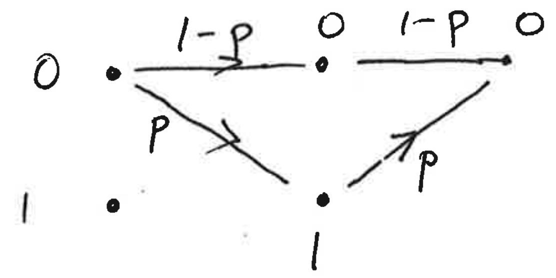
$$\therefore f''(x) = \sum_{n=1}^{\infty} 4 \cos n\pi x$$

(d) Integrating $\quad f'(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin n\pi x + \text{const}$ $\quad$ & d.c. value for $f' = 0 \Rightarrow \text{const} = 0$

" $\quad f(x) = -\sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x + d$ $\quad$ & d as above

8

(a) By symmetry probability of returning to 0 if it starts in 0 is the same as that of returning to 1 if it starts in 1. Therefore, only need to consider one of these.

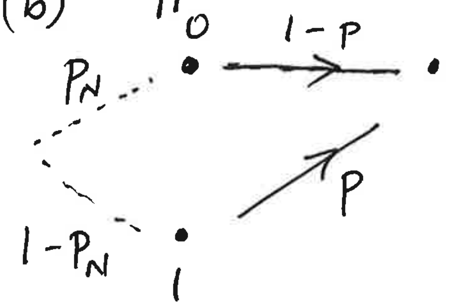


Bit remains at 0 for 2 cycles with prob $(1-p)^2$

Bit transitions to 1 & returns with prob p^2

$$\therefore \text{Probability} = p^2 + (1-p)^2$$

(b) Suppose started at 0



$$P_{N+1} = (1-p)P_N + p(1-P_N)$$

$$= p + (1-2p)P_N$$

9. $$f(x,y) = x^2 + 4xy + 2y^2 - 6x - 8y$$

$$\frac{\partial f}{\partial x} = 2x + 4y - 6$$

$$\frac{\partial f}{\partial y} = 4x + 4y - 8$$

At a stationary pt $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$$\Rightarrow \quad \begin{aligned} x + 2y &= 3 \\ 2x + 2y &= 4 \end{aligned} \quad \Rightarrow \quad \underline{x = 1, \ y = 1}$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 4 \qquad \frac{\partial^2 f}{\partial y^2} = 4$$

So $\frac{\partial^2 f}{\partial x^2}$ & $\frac{\partial^2 f}{\partial y^2} > 0$

& $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2.4 - 4^2 < 0$

$\Rightarrow$ $\underline{(1,1) \text{ is a saddle pt}}$

10.

(a) L.T. of $f(t)\,e^{-at} = \int_{t=0}^{\infty} f(t)\,e^{-at-st}\,dt$

& L.T. of $f(t) = \int_{t=0}^{\infty} f(t)\,e^{-st}\,dt = F(s)$

$\therefore$ L.T. of $f(t)\,e^{-at}$ is $\underline{F(s+a)}$

(b) Taking L.T.

$$s\,Y(s) - y(0) + Y(s) = \frac{1}{(s+1)^2+1}$$

$$\therefore \quad Y = \left\{1 + \frac{1}{(s+1)^2+1}\right\}\frac{1}{s+1}$$

$$= \frac{s^2+2s+3}{(s^2+2s+2)(s+1)}$$

$$= \frac{As+B}{s^2+2s+2} + \frac{C}{s+1}$$

$$\therefore \quad s^2+2s+3 = (As+B)(s+1) + C(s^2+2s+2)$$

Equating coefficients

s^2: $\quad 1 = A + C \quad$ (a)

s^1: $\quad 2 = A + B + 2C \quad$ (b)

s^0: $\quad 3 = B + 2C \quad$ (c)

Subtracting (b) from (c) $\Rightarrow 1 = -A \Rightarrow C = 2$ & $B = -1$

$$\therefore \quad Y = \frac{-(s+1)}{(s+1)^2+1} + \frac{2}{s+1}$$

$$\Rightarrow \quad \underline{y(t) = 2e^{-t} - e^{-t}\cos t}$$

(c) $L(f*g) = F(s)\,G(s)$

Taking $F(s) = G(s) = \dfrac{1}{s^2+1} \Rightarrow f(t) = g(t) = \sin t$

then inverse transform $= \displaystyle\int_{\tau=0}^{t} f(\tau)\,g(t-\tau)\,d\tau$

$$= \int_{\tau=0}^{t} \sin\tau \sin(t-\tau)\,d\tau = \frac{1}{2}\int_{\tau=0}^{t} \left[\cos(2\tau-t) - \cos t\right] d\tau$$

$$= -\frac{\cos t}{2}\left[\tau\right]_0^t + \frac{1}{2}\left[\frac{\sin(2\tau-t)}{2}\right]_0^t$$

$$= \frac{1}{4}\left[\sin t - \sin(-t)\right] - \frac{t\cos t}{2}$$

$$= \frac{1}{2}(\sin t - t\cos t)$$

(d) Taking L.T.

$$s^2 Y - s\underset{0}{\underset{\shortparallel}{y_0}} - \underset{0}{\underset{\shortparallel}{\dot{y}_0}} + 2(sY - \underset{0}{\underset{\shortparallel}{y_0}}) + 2Y = \frac{1}{(s+1)^2+1}$$

$$\Rightarrow \quad Y = \frac{1}{\left[(s+1)^2+1\right]^2}$$

which appeared in part (c) but $s \to s+1$

$$\therefore \quad y(t) = \frac{e^{-t}}{2}(\sin t - t\cos t)$$

Crib 1

(a) The `bool` value indicates whether the quadratic equation has real roots or not. If not, the solutions in `s1` and `s2` are invalid. [2]

(b) The `&` signs indicate *call be reference* as opposed to *call by value*. If they were omitted, the function would store the two roots in local copies of `s1` and `s2` instead of the actual parameters passed by the calling routine. The calling routine would have no way of accessing the roots. [3]

(c) This is *subtractive cancellation*. When we subtract two similar numbers `a` and `b`, the intended result may be of comparable magnitude to the truncation errors when storing `a` and `b` to finite precision, and therefore subject to significant error. We could correct the error by changing the variable types from `float` to `double` and/or obtaining `s1` instead using the known product of the roots: `s1 = c/(a*s2)`. [5]

Crib 2

(a) The first code segment calculates the transpose of the $n \times n$ matrix `m1`, storing the result in the $n \times n$ matrix `m2`. It has complexity $O(n^2)$. The second code segment calculates the sum of the n-element vectors `v1` and `v2`, storing the result in the n-element vector `v3`. It has complexity $O(n)$. The third code segment calculates the product of the $n \times n$ matrices `m2` and `m1` (in the order `m2*m1`), storing the result in the $n \times n$ matrix `m3`. It has complexity $O(n^3)$. [7]

(b) The key point here is to ignore the first two code segments, whose execution times are negligible compared with the third. Given the cubic complexity, when `n` doubles we would expect the execution time to increase by a factor of eight to 8 s. (Most likely it will take even longer than this, because of cache misses, but Part IA students are not expected to appreciate this subtlety.) [3]