EGT0
ENGINEERING TRIPOS PART IA

Tuesday 10 June 2014 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

1 (**short**) Find all of the possible values of the determinant of the matrix **A** given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$$

where $a^3 = 1$. [10]

2 (**short**) Sketch carefully the curve $y = \frac{1}{x} - \cos x$ for $-2\pi < x < 2\pi$, marking on your sketch the approximate positions of the two intersections with the x axis that are nearest to $x = 0$. [10]

3 (**short**)

(a) A real, symmetric 3×3 matrix **A** has eigenvectors $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^t$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^t$ and corresponding eigenvalues 3 and 2. If the third eigenvalue is 1, find such a matrix **A**. [6]

(b) Describe two simple checks that could help verify that your answer is correct. [4]

4 (**long**)

(a) Find the point of intersection P of the line $\underline{r} = (1,3,0) + \lambda\,(1,-1,1)$ and the plane $z = 2y$. [6]

(b) The volume of a tetrahedron is given by $\frac{1}{3} \times \text{base area} \times \text{height}$. Show that the volume can be written as $\frac{1}{6}$ of the magnitude of the scalar triple product of the vectors representing the edges of the tetrahedron that meet at a vertex. [10]

(c) Find the volume of the tetrahedron whose vertices are: the origin, the point P found in part (a), the point (1,3,0) and the point $(\alpha,0,0)$. [4]

(d) Find the shortest distance from the point (1,3,0) to the opposite face of the tetrahedron defined in part (c). [10]

5 (**long**) The differential equation governing the output voltage V of a circuit subject to a switching input is

$$\frac{d^2V}{dt^2} + 4V = \begin{cases} 1 & 0 < t \le \pi/2 \\ 0 & t > \pi/2 \end{cases}$$

where the voltage V and dV/dt are continuous at the point of switching and where $V = 0$ and $dV/dt = 0$ at $t = 0$.

(a) Find V for t in the range $0 < t < \pi/2$. [5]

(b) Find V for all $t \ge \pi/2$. [6]

(c) If the right hand side of the differential equation is changed to 0.1 for $t > \pi/2$, find and sketch $V(t)$. [19]

SECTION B

6 (**short**) The output of a linear system, $y(t)$, is governed by the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = f(t)$$

where $f(t)$ is the input. Find and sketch the impulse response of the system. [10]

7 (**long**) The function $f(x)$ is periodic with period 2 and is given by

$$f(x) = 2x - x^2 \quad \text{for} \quad 0 \le x \le 2$$

(a) Show that f is an even function of x and explain the implications of this for a Fourier Series representation of f. [5]

(b) Find a Fourier Series representation of f, evaluating the coefficients by direct integration. [10]

(c) Show that, for $-1 < x < 1$,

$$\frac{d^2 f}{dx^2} = 4\delta(x) - 2$$

and find a Fourier Series representation for $\frac{d^2 f}{dx^2}$, evaluating the coefficients by direct integration. [5]

(d) Using the result obtained in part (c), find a Fourier Series representation for f and verify that it agrees with the result obtained in part (b). [10]

8 (**short**) A faulty bit in a computer memory changes state in any given clock cycle with probability p.

(a) Find the probability that it will be in the same state two clock cycles later. [5]

(b) If P_N is the probability that it is in the same state N cycles later, show that

$$P_{N+1} = p + (1-2p) P_N$$ [5]

9 (**short**) The function $f(x, y)$ is given by

$$f(x,y) = x^2 + 4xy + 2y^2 - 6x - 8y$$

Find and classify the stationary points of f . [10]

10 (**long**)

(a) Show that the Laplace Transform of $f(t)\,e^{-at}$ is $F(s+a)$, where $F(s)$ is the Laplace Transform of $f(t)$ and where a is a constant. [3]

(b) Using Laplace Transforms, and no other method, solve the differential equation

$$\frac{dy}{dt} + y = e^{-t}\sin t$$

subject to the boundary condition $y(0) = 1$. [11]

(c) Using the relationship that relates the Laplace Transform of the convolution of two functions

$$f * g = \int_{\tau=0}^{t} f(\tau)\,g(t-\tau)\,d\tau$$

to those of the individual functions, find the inverse transform of

$$\frac{1}{\left(s^2+1\right)^2}$$ [11]

(d) Hence find the solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^{-t}\sin t$$

subject to the boundary conditions $y = 0$ and $dy/dt = 0$ at $t = 0$. [5]

SECTION C

11 (**short**) Consider the following C++ function for solving a quadratic equation:

```
bool solve(float a, float b, float c, float &s1, float &s2)
{
  float d = b*b - 4*a*c;
  if (d<0) return false;
  else {
    s1 = (-b + sqrt(d)) / (2.0*a);
    s2 = (-b - sqrt(d)) / (2.0*a);
    return true;
  }
}
```

(a) Explain what is indicated by the `bool` value that the function returns. [2]

(b) Explain the purpose of the two & signs in the parameter list. How would the program malfunction if they were omitted? [3]

(c) When `a`, `b` and `c` are 0.011, 64.1 and 0.011 respectively, the function calculates `s1` as -3.47×10^{-4}, whereas the correct solution is -1.72×10^{-4}. Explain the cause of this large error and suggest how it might be corrected. [5]

12 (**short**) Consider the following three C++ code segments.

```
// Segment 1
for (i=0; i<n; i++)
  for (j=0; j<n; j++)
    m2[j][i] = m1[i][j];

// Segment 2
  for (i=0; i<n; i++)
    v3[i] = v2[i] + v1[i];

// Segment 3
for (i=0; i<n; i++)
  for (j=0; j<n; j++) {
    m3[i][j] = 0;
    for (k=0; k<n; k++)
      m3[i][j] += m2[i][k] * m1[k][j];
  }
```

(a) For each code segment, state

(i) its purpose and

(ii) its algorithmic complexity. [7]

(b) All three segments are executed sequentially, taking a total time of 1 s when `n` is 1000. Estimate how long, in total, they would take to execute when `n` is 2000. [3]

END OF PAPER

Answers

1 $0,\ \pm 3\sqrt{3}\, i$

2 Intersections at –2.07, –4.49

3 (a) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

4 (a) $(\ 3, 1, 2\ )$ (c) α (d) $\dfrac{6}{\sqrt{5}}$

5 (a) $\dfrac{1}{4}\left(1-\cos 2t\right)$ (b) $-\dfrac{1}{2}\cos 2t$ (c) $\dfrac{1}{40}\left(1-19\cos 2t\right)$

6 $t\,e^{-t}$

7 (b) $\dfrac{2}{3} - \displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x$ (c) $\displaystyle\sum_{n=1}^{\infty} 4\cos n\pi x$

8 (a) $1-2p+2p^2$

9 (1,1) , saddle

10 (b) $2e^{-t} - e^{-t}\cos t$ (c) $\dfrac{1}{2}\left(\sin t - t\cos t\right)$ (d) $\dfrac{e^{-t}}{2}\left(\sin t - t\cos t\right)$

12 (b) $8\ s$