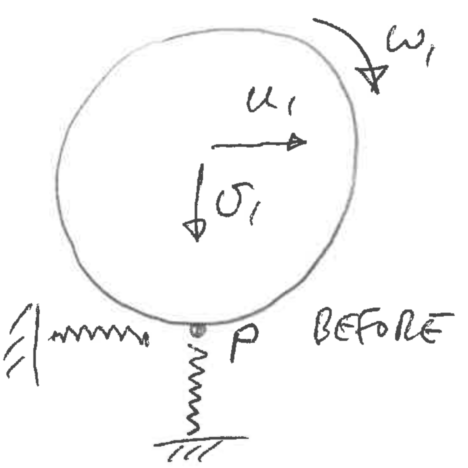
Part IB Mechanics – Paper 1   2018   1

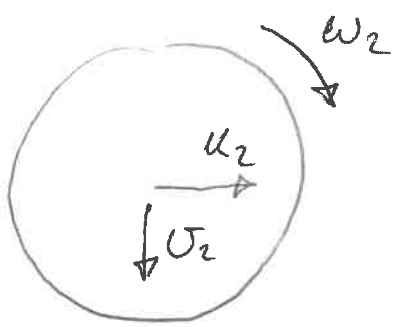
(This is a crib provided by Hugh Hunt who was not one of the examiners)
(No guarantee of accuracy – it has not been checked)

1.




Consider a contact particle that is completely elastic and the body is rigid. Assume that vertical and horizontal "springs" are independent

Elastic motion in the vertical direction, assume that $v_2 = -v_1$




<u>Conservation of KE</u>

(a)
$$\tfrac{1}{2} m u_1^2 + \tfrac{1}{2} m v_1^2 + \tfrac{1}{2} I \omega_1^2 = \tfrac{1}{2} m u_2^2 + \tfrac{1}{2} m v_2^2 + \tfrac{1}{2} I \omega_2^2$$

Use $I = \frac{2}{5} m a^2$

$$\therefore \quad u_1^2 - u_2^2 = \frac{2}{5} a^2 \left( \omega_2^2 - \omega_1^2 \right)$$

and use "difference of two squares"

$$\therefore \quad (u_1 - u_2)(u_1 + u_2) = \frac{2}{5} a^2 (\omega_2 - \omega_1)(\omega_2 + \omega_1) \quad 1$$

Conservation of moment of momentum about P

(note that all impulse forces pass through P)

$$m u_1 a + I\omega_1 = m u_2 a + I\omega_2$$

$$\therefore \quad u_1 - u_2 = \tfrac{2}{5} a (\omega_2 - \omega_1) \qquad (2)$$

which is ans to (a)

(b)

substitute into (1) ~~after~~

$$\therefore \quad \tfrac{2}{5} a(\omega_2 - \omega_1)(u_1 + u_2) = \tfrac{2}{5} a^2(\omega_2 - \omega_1)(\omega_2 + \omega_1)$$

$$\therefore \quad u_1 + u_2 = a(\omega_1 + \omega_2) \qquad (3)$$

(2) + (3)

$$\therefore \quad 2u_1 = a\left(\left(\tfrac{2}{5}+1\right)\omega_2 + \left(1 - \tfrac{2}{5}\right)\omega_1\right)$$

$$= a\left(\tfrac{7}{5}\omega_2 + \tfrac{3}{5}\omega_1\right)$$

$$\therefore \quad \omega_2 = \frac{1}{7}\left(\frac{10 u_1}{a} - 3\omega_1\right)$$

substitute into (3)

$$\therefore \quad u_1 + u_2 = a\omega_1 + \frac{10 u_1}{7} - \frac{3a\omega_1}{7}$$

$$\therefore \quad u_2 - \tfrac{3}{7} u_1 = \tfrac{4}{7} a\omega_1$$

$$\therefore \quad u_2 = \tfrac{3}{7} u_1 + \tfrac{4}{7} a\omega_1$$

ans

(c) if ball bounces straight up

$\therefore u_2 = 0 \quad \therefore \quad \omega_1 = -\frac{3u_1}{4a}$ ans

Does this make sense? Yes, you need "back spin" to make the ball return to the thrower and to "sit up".

Out of interest, what is $\omega_2$?

$$\omega_2 = \frac{1}{7}\left(\frac{10u_1}{a} - 3\omega_1\right)$$

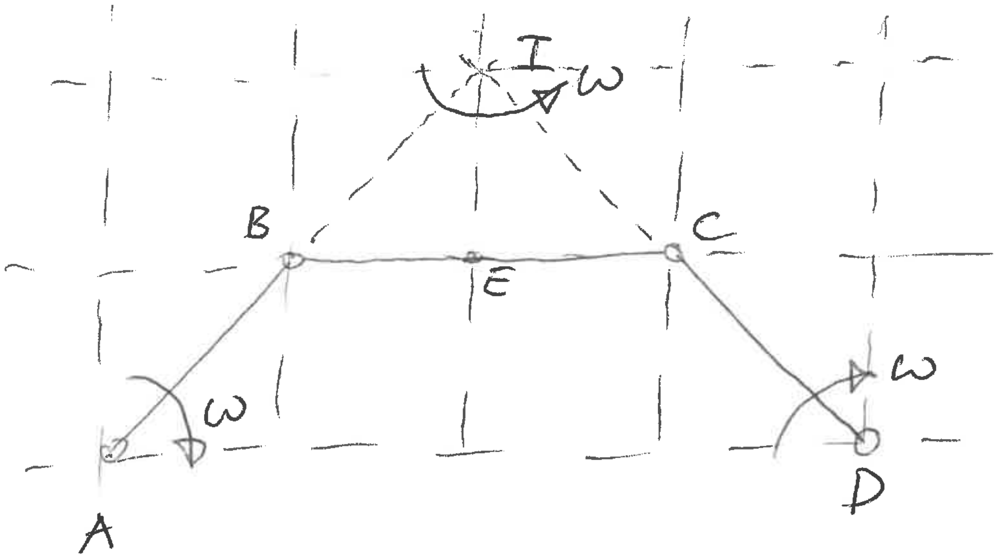
$$= \frac{1}{7}\left(\frac{10u_1}{a} + \frac{9u_1}{4a}\right)$$

$$= \frac{7}{4a}u_1 \qquad = -\frac{7}{3}\omega_1$$

Note that spin direction reverses. Try it!
There are videos at www2.eng.cam.ac.uk/~hemh1/movies.htm#superballs

(There was a similar exam question set a few years ago. I think the assumptions made there were wrong. The assumptions here are consistent with observations)

2 (a) Use instant centres




$\omega_{CD} = \omega$ ↻ ans

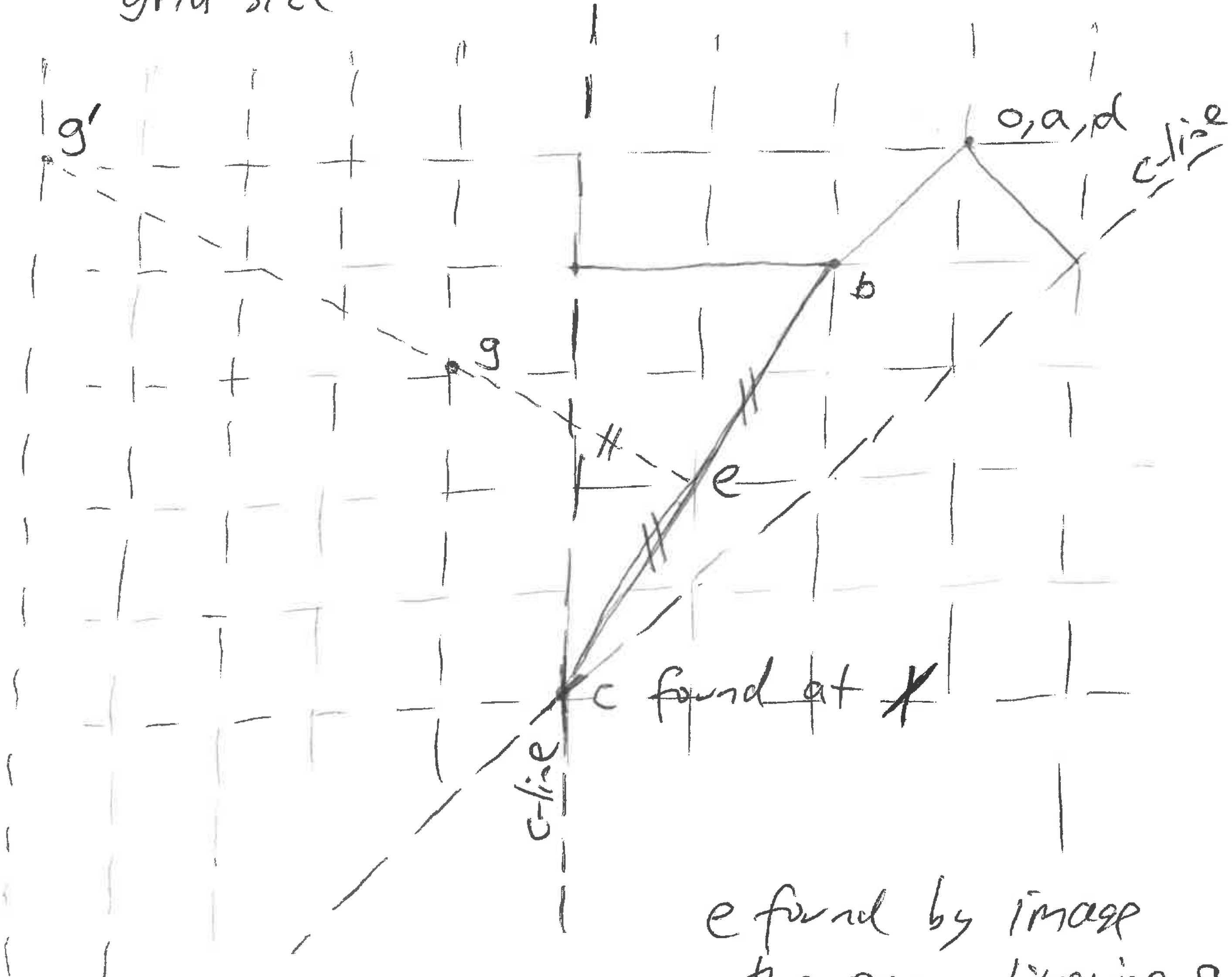
(b) This can be done either by acceleration diagram or by vectors attached to the members

I'll do both !

Acceleration diagram    tabulate info for diagram:

| member | $\underline{e}$ | $\underline{e}^*$ | $(\ddot{r} - r\dot{\theta}^2)\underline{e}$ | $(2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}^*$ |
|---|---|---|---|---|
| AB | ↗ | ↘ | $0 - a\sqrt{2}\omega^2$ | 0 |
| BC | → | ↑ | $0 - 2a\omega^2$ | 0 + ? |
| CD | ↘ | ↙ | $0 - a\sqrt{2}\omega^2$ | 0 + ? |

grid size $= a\omega^2$




e found by image theorem, likewise g

$\therefore$ acceleration of e

$= \sqrt{(3a\omega^2)^2 + (2a\omega^2)^2}$

$= \sqrt{13}\, a\omega^2$ ans (b)

and acceleration of g is $\sqrt{2^2+4^2}\, a\omega^2$

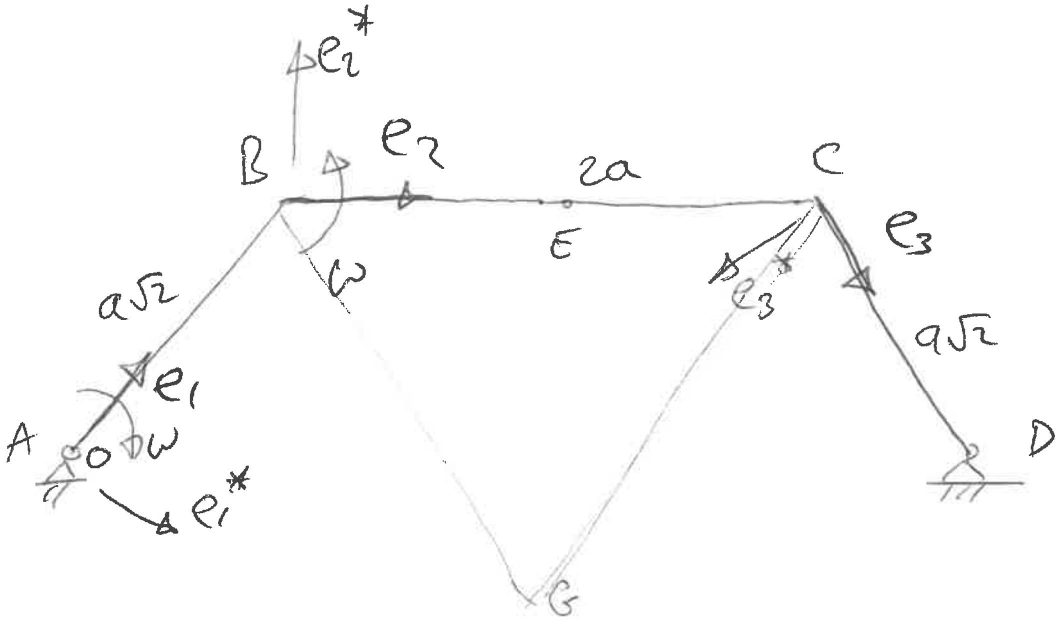
$= 5\sqrt{2}\, a\omega^2$ ans (c)

(d) Want g not to have downward acceleration

$\therefore$ g at g' $\therefore$ $d = 3a$ ans

This is so that centre of curvature of path of G is above G

Now do it using unit vectors as follows




$$\underline{r}_D = a\sqrt{2}\,\underline{e}_1 + 2a\,\underline{e}_2 + a\sqrt{2}\,e_3$$

$$\dot{\underline{r}}_D = a\sqrt{2}\,\omega_1 e_1^* + 2a\omega_2 \underline{e}_2^* + a\sqrt{2}\,\omega_3 e_3^*$$

$$\ddot{\underline{r}}_D = -a\sqrt{2}\,\omega_1^2 \underline{e}_1 - 2a\omega_2^2 e_2 - a\sqrt{2}\,\omega_3^2 \underline{e}_3$$

$$+ 2a\dot{\omega}_2 e_2^* + a\sqrt{2}\,\dot{\omega}_3 e_3^* = 0$$

since D is fixed

$\ddot{\underline{r}}_D \cdot \underline{e}_2$ to eliminate $\dot{\omega}_2$

and note $\omega_1 = \omega_2 = \omega_3 = \omega$

$$\therefore\ -a\sqrt{2}\,\omega^2(\underline{e}_1 \cdot \underline{e}_2) - 2a\omega^2(\underline{e}_2 \cdot \underline{e}_2) - a\sqrt{2}\,\omega^2(\underline{e}_3 \cdot \underline{e}_2)$$

$$+ 2a\dot{\omega}_2(\underline{e}_2^* \cdot \underline{e}_2) + a\sqrt{2}\,\dot{\omega}_3(e_3^* \cdot \underline{e}_2) = 0$$

$$\therefore\ -a\sqrt{2}\,\omega^2 \frac{\sqrt{2}}{2} - 2a\omega^2 - a\sqrt{2}\,\omega^2 \frac{\sqrt{2}}{2}$$

$$+ 0 + a\sqrt{2}\,\dot{\omega}_3\left(-\frac{\sqrt{2}}{2}\right) = 0$$

$$\therefore\ \dot{\omega}_3 = -4\omega^2$$

(actually I didn't need to know $\dot{\omega}_3$ !)

$\ddot{\underline{r}}_D \cdot \underline{e}_3$ to eliminate $\dot{\omega}_3$

$$\therefore \; -a\sqrt{2}\,\omega^2(\underline{e}_1\cdot\underline{e}_3) - 2a\omega^2(\underline{e}_2\cdot\underline{e}_3) - a\sqrt{2}\,\omega^2(\underline{e}_3\cdot\underline{e}_3)$$
$$+ 2a\dot{\omega}_2(\underline{e}_2^{*}\cdot\underline{e}_3) + a\sqrt{2}\,\dot{\omega}_3(\underline{e}_3^{*}\cdot\underline{e}_3) = 0$$

$$\therefore \; -a\sqrt{2}\,\omega^2(0) - 2a\omega^2\frac{\sqrt{2}}{2} - a\sqrt{2}\,\omega^2$$
$$+ 2a\dot{\omega}_2\left(-\frac{\sqrt{2}}{2}\right) = 0$$

$$\therefore \; \dot{\omega}_2 = -2\omega^2$$

(which is in agreement with acceleration diagram)

Now acceleration of E

$$\underline{r}_E = a\sqrt{2}\,\underline{e}_1 + a\,\underline{e}_2$$

$$\dot{\underline{r}}_E = a\sqrt{2}\,\omega_1\,\underline{e}_1^{*} + a\omega_2\,\underline{e}_2^{*}$$

$$\ddot{\underline{r}}_E = -a\sqrt{2}\,\omega_1^2\,\underline{e}_1 - a\omega_2^2\,\underline{e}_2 + a\dot{\omega}_2\,\underline{e}_2^{*}$$

$$= -a\sqrt{2}\,\omega^2\,\underline{e}_1 - a\omega^2\,\underline{e}_2 - 2a\omega^2\,\underline{e}_2^{*}$$

$$= -a\omega^2\left(\sqrt{2}\,\frac{\underline{i}+\underline{j}}{\sqrt{2}} + \underline{i} + 2\underline{j}\right)$$

$$= -a\omega^2(2\underline{i} + 3\underline{j})$$

$$|\ddot{\underline{r}}_E| = \sqrt{13}\,a\omega^2$$ as before  ans (6)

Note how long winded this is !

Now G :

$$\underline{r}_G = a\sqrt{2}\,\underline{e}_1 + a\,\underline{e}_2 - d\,\underline{e}_2^*$$

$$\dot{\underline{r}}_G = a\sqrt{2}\,\omega_1\,e_1^* + a\omega_2\,e_2^* + d\omega_2\,e_2$$

$$\ddot{\underline{r}}_G = -a\sqrt{2}\,\omega_1^2\,\underline{e}_1 - a\omega_2^2\,e_2 + d\omega_2^2\,e_2^*$$

$$+ a\dot{\omega}_2\,\underline{e}_2^* + d\dot{\omega}_2\,\underline{e}_2$$

$$= -a\omega^2\left(\sqrt{2}\,\underline{e}_1 + \underline{e}_2 - \frac{d}{a}\,\underline{e}_2^* + 2\,\underline{e}_2^* + \frac{2d}{a}\,\underline{e}_2\right)$$

$$= -\omega^2\left(a\sqrt{2}\,\underline{e}_1 + (a+2d)\,\underline{e}_2 + (-d+2a)\,\underline{e}_2^*\right)$$

$$= -\omega^2\left(a\sqrt{2}\,\frac{\underline{i}+\underline{j}}{\sqrt{2}} + (a+2d)\,\underline{i} + (-d+2a)(-\underline{j})\right)$$

$$= -\omega^2\left(a(\underline{i}+\underline{j}) + (a+2d)\,\underline{i} + (d+2a)\,\underline{j}\right)$$

$$= -\omega^2\left(2(d+a)\,\underline{i} + (3a-d)\,\underline{j}\right)$$

(c) if $d = a$ $\therefore$ $\ddot{\underline{r}}_G = -\omega^2(4\underline{i} + 2\underline{j})a$

as on acceleration diagram

(d) For $\ddot{\underline{r}}_G$ to have no $\underline{j}$ component

then $3a - d = 0$ $\therefore$ $\underline{\underline{d = 3a}}$

as before

Note that this question is not appropriate for the new syllabus that began in 2018-19 because solution of mechanisms by acceleration diagram or vector method is not covered in this level of detail

Note on part (d)

For $d < 3a$

path of G is unstable

because acceleration of G is $\uparrow$

for $d = 3a$

path of G is

because acceleration of G has no $\underline{j}$ component

for $d > 3a$ path of G is

which is stable

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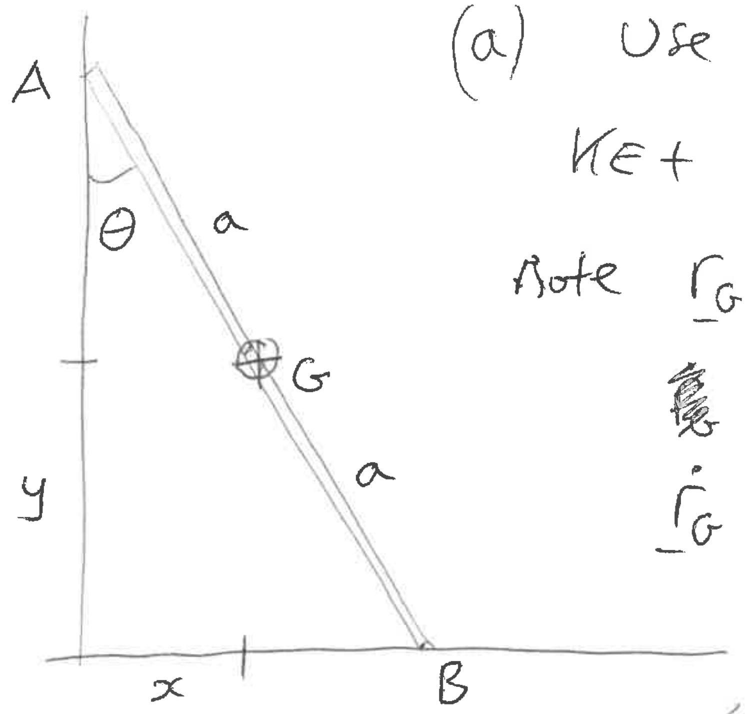



(a) Use energy

$$KE + PE = \text{const}$$

Note $\underline{r}_G = x\underline{i} + y\underline{j}$

$$= a\sin\theta\,\underline{i} + a\cos\theta\,\underline{j}$$

$$\dot{\underline{r}}_G = a\dot{\theta}\cos\theta\,\underline{i} - a\dot{\theta}\sin\theta\,\underline{j}$$

$$KE = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\dot{\theta}^2$$

$$= \tfrac{1}{2}m\left((a\dot{\theta}\cos\theta)^2 + (a\dot{\theta}\sin\theta)^2\right)$$

$$+ \tfrac{1}{2}\,\tfrac{1}{3}ma^2\dot{\theta}^2$$

$$= \tfrac{1}{2}m(a\dot{\theta})^2 + \tfrac{1}{2}\,\tfrac{1}{3}m(a\dot{\theta})^2$$

$$= \tfrac{2}{3}m(a\dot{\theta})^2$$

$$PE = mga(1-\cos\theta)$$

datum $PE = 0$ at $\theta = 0$ just for neatness

$$PE + KE = 0$$

$$\therefore \tfrac{2}{3}(a\dot{\theta})^2 = ga(1-\cos\theta)$$

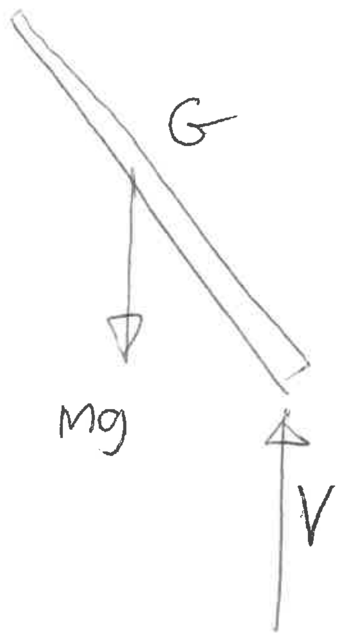
$$\therefore \dot{\theta}^2 = \tfrac{3}{2}\,\tfrac{g}{a}(1-\cos\theta)$$

Now use $\dfrac{d}{d\theta}\left(\tfrac{1}{2}\dot{\theta}^2\right) = \ddot{\theta}$

$$\therefore \quad \ddot{\theta} = \frac{1}{2}\frac{d}{d\theta}\left(\frac{3}{2}\frac{g}{a}(1-\cos\theta)\right)$$

$$\ddot{\theta} = \frac{3}{4}\frac{g}{a}\sin\theta \qquad // \ \underline{\text{ans}}$$

(b) When ladder loses contact there is no horizontal force on the ladder $\therefore \ \underline{\ddot{r}_G} \cdot \underline{i} = 0$




$$\underline{\ddot{r}_G} = a\ddot{\theta}\cos\theta\,\underline{i} - a\ddot{\theta}\sin\theta\,\underline{j} - a\dot{\theta}^2\sin\theta\,\underline{i} - a\dot{\theta}^2\cos\theta\,\underline{j}$$

so for $\underline{\ddot{r}_G} \cdot \underline{i} = 0$

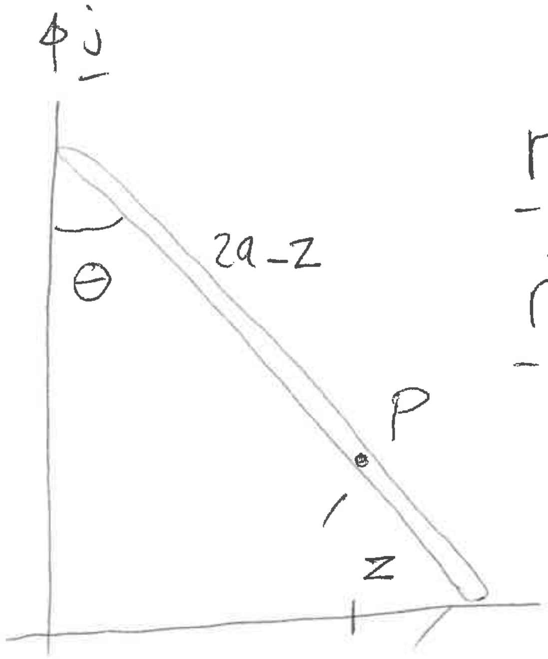
$$\ddot{\theta}\cos\theta = \dot{\theta}^2\sin\theta$$

$$\therefore \ \frac{3}{4}\frac{g}{a}\sin\theta\cos\theta = \frac{3}{2}\frac{g}{a}\sin\theta(1-\cos\theta)$$

$$\therefore \ \frac{3}{4}\cos\theta = \frac{3}{2}(1-\cos\theta)$$

$$\therefore \ \cos\theta + 2\cos\theta = 2$$

$$\therefore \quad \cos\theta = \frac{2}{3} \qquad \underline{\underline{\text{ans}}}$$




$$\underline{r}_p = (2a - z)\sin\theta\,\underline{i} + z\cos\theta\,\underline{j}$$

$$\underline{\dot r}_p = (2a - z)\dot\theta\cos\theta\,\underline{i} - z\dot\theta\sin\theta\,\underline{j}$$

vertical component of $\underline{\ddot r}_p = \left(-z\ddot\theta\sin\theta - z\dot\theta^2\cos\theta\right)\underline{j}$

$$= -z\left(\frac{3}{4}\frac{g}{a}\sin^2\theta + \frac{3}{2}\frac{g}{a}(1-\cos\theta)\cos\theta\right)\underline{j}$$

$$= -z\frac{g}{a}\left(\frac{3}{4}\sin^2\theta + \frac{3}{2}\cos\theta - \frac{3}{2}\cos^2\theta\right)$$

Presume answer wanted at the instant of separation $\therefore$ $\cos\theta = \frac{2}{3}$

$\cos^2\theta = \frac{4}{9}$

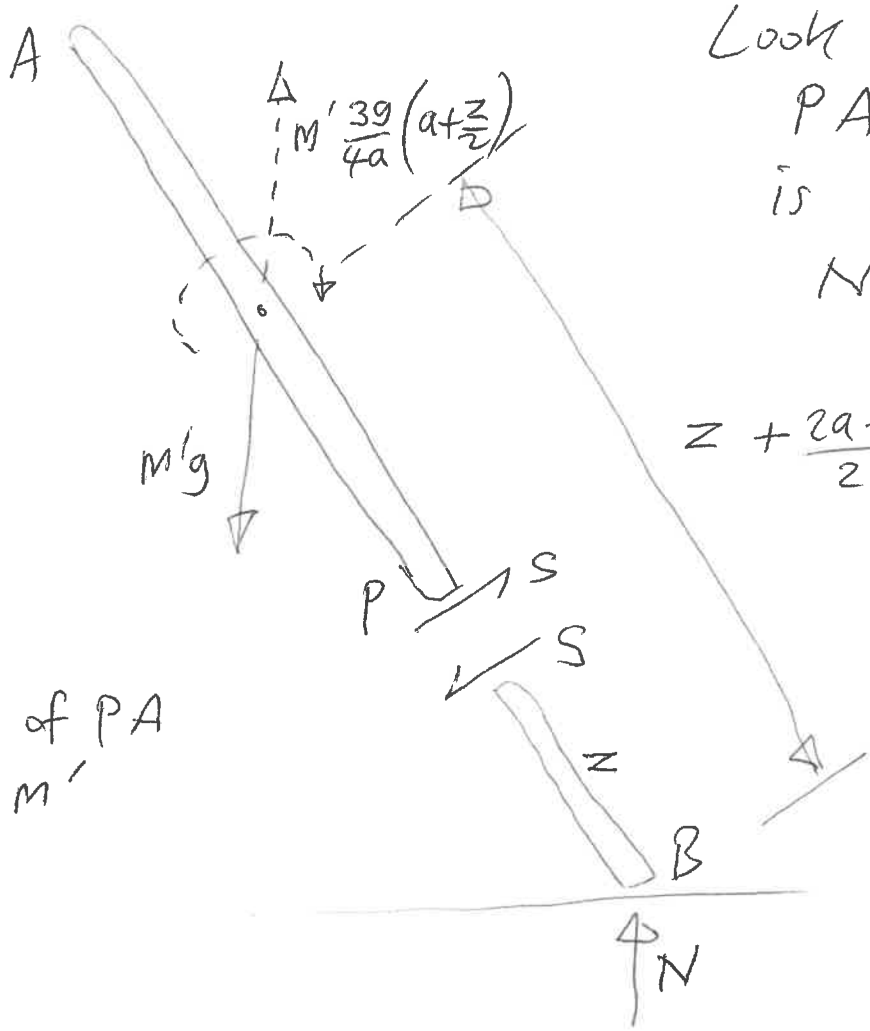
$\sin^2\theta = \frac{5}{9}$

$$\therefore\ accel_v = -z\frac{g}{a}\left(\frac{3}{4}\frac{5}{9} + \frac{3}{2}\frac{2}{3} - \frac{3}{2}\frac{4}{9}\right)$$

$$= -\frac{zg}{3a}\left(\frac{5}{4} + 3 - 2\right)$$

$$= -\frac{zg}{3a}\frac{9}{4} = \frac{-3zg}{4a} \quad \checkmark \ \underline{\underline{\text{ans}}}$$

(d) Greatest BM is where shear is zero because $S = \frac{dM}{dx}$




Look at top section PA because there is no normal reaction N

$z + \frac{2a-z}{2} = a + \frac{z}{2}$

Mass of PA is $M'$

For $S = 0$ then resolve normal to AP

$\therefore M'g = M'\frac{3g}{4a}\left(a+\frac{z}{2}\right)$

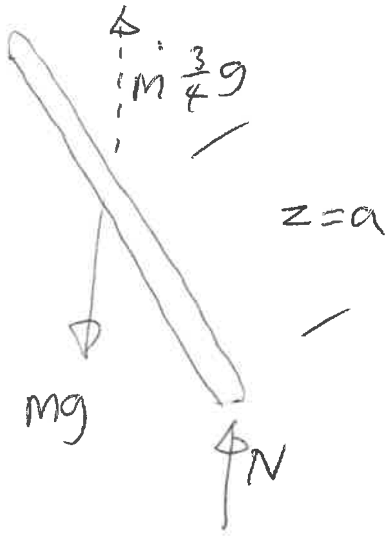
$\therefore \frac{4a}{3} = a + \frac{z}{2} \quad \therefore \underline{\underline{z = \frac{2a}{3}}}$ ans

~~Check using bottom half~~

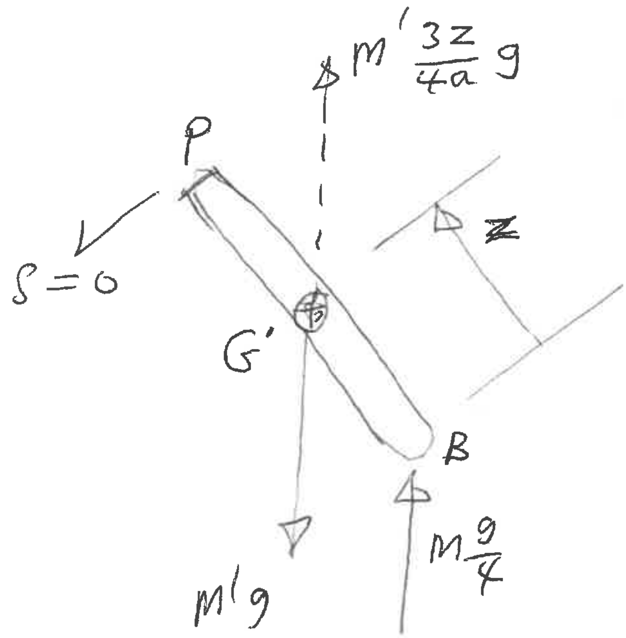
Check by using section BP

First find $N$ for whole of $AB$ with $z = a$

so vertical accel of $G = \frac{3}{4}g$




$\therefore N = m\frac{g}{4}$




note mass of $BP$ length $2z$

$$m' = \frac{2z}{2a}m = \frac{z}{a}m$$

Resolve $\perp$ BP

$$\therefore m'g = m'\frac{3z}{4a}g + m\frac{g}{4}$$

$$\therefore \frac{z}{a}\left(1 - \frac{3}{4}\frac{z}{a}\right) = \frac{1}{4}$$

Note here that $z$ is the distance to $G'$

so substitute $z = \frac{a}{3}$ from previous page

$$LHS = \frac{1}{3}\left(1 - \frac{3}{4}\frac{1}{3}\right) = \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4}$$

$= RHS$ ✓

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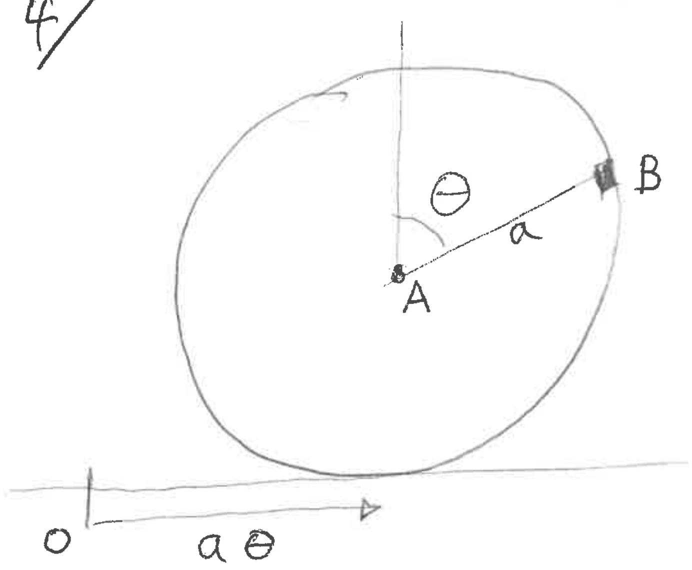



(a) position of centre of hoop, rolls distance $a\theta$

$$\underline{r}_A = a\theta\,\underline{i} + a\,\underline{j}$$

velocity of A : $\dot{\underline{r}}_A = a\dot{\theta}\,\underline{i}$

position of mass B

$$\underline{r}_B = \underline{r}_A + a\sin\theta\,\underline{i} + a(1+\cos\theta)\,\underline{j}$$

$$\dot{\underline{r}}_B = \dot{\underline{r}}_A + a\dot{\theta}\cos\theta\,\underline{i} - a\dot{\theta}\sin\theta\,\underline{j}$$

$$= a\dot{\theta}(1+\cos\theta)\,\underline{i} - a\dot{\theta}\sin\theta\,\underline{j}$$

<u>check</u> : $\theta = \pi$ $\quad \therefore \dot{\underline{r}}_B = a\dot{\theta}(1-1)\,\underline{i} - a\dot{\theta}\,0\,\underline{j}$

$$= 0$$

This is correct, when B is at the contact point.

$$KE = \tfrac{1}{2} m v_A^2 + \tfrac{1}{2} I \dot{\theta}^2 + \tfrac{1}{2} m v_B^2$$

$$I = ma^2$$

$$\therefore KE = \tfrac{1}{2} m (a\dot{\theta})^2 + \tfrac{1}{2} ma^2 \dot{\theta}^2 \cancel{+\tfrac{1}{2}m}$$

$$+ \tfrac{1}{2} m \left[ (a\dot{\theta}(1+\cos\theta))^2 + (a\dot{\theta}\sin\theta)^2 \right]$$

$$= \tfrac{1}{2} m (a\dot{\theta})^2 \left[ 1 + 1 + 1 + 2\cos\theta + \cos^2\theta + \sin^2\theta \right]$$

$$= \tfrac{1}{2} m (a\dot{\theta})^2 \left[ 4 + 2\cos\theta \right]$$

$$PE = mga(1+\cos\theta)$$

$$PE + KE = \text{const}$$

$$\therefore \tfrac{1}{2} m (a\dot{\theta})^2 [4 + 2\cos\theta] + mga(1+\cos\theta) = 2mga$$

$$\therefore \dot{\theta}^2 = \frac{g}{a} \frac{(1-\cos\theta)}{2+\cos\theta}$$ ans ✓

(b) Use $$\ddot{\theta} = \frac{d}{d\theta}\left(\tfrac{1}{2}\dot{\theta}^2\right)$$

$$= \frac{g}{2a} \frac{\sin\theta(2+\cos\theta) + \sin\theta(1-\cos\theta)}{(2+\cos\theta)^2}$$

and at $\theta = \frac{\pi}{2}$ $\quad \sin\theta = 1 \quad \cos\theta = 0$

$$\therefore \ddot{\theta} = \frac{g}{2a}\left(\frac{2+1}{4}\right)$$

$$= \frac{3g}{8a} \quad \checkmark \qquad \text{ans}$$

(c) d'Alembert : need accelerations

$$\ddot{\underline{r}}_A = a\ddot{\theta}\,\underline{i}$$

$$\ddot{\underline{r}}_B = a\ddot{\theta}(1+\cos\theta)\,\underline{i} + a\dot{\theta}(-\dot{\theta}\sin\theta)\,\underline{i} - a\ddot{\theta}\sin\theta\,\underline{j} - a\dot{\theta}^2\cos\theta\,\underline{j}$$

at $\theta = \frac{\pi}{2}$ $\quad \ddot{\theta} = \frac{3g}{8a}$

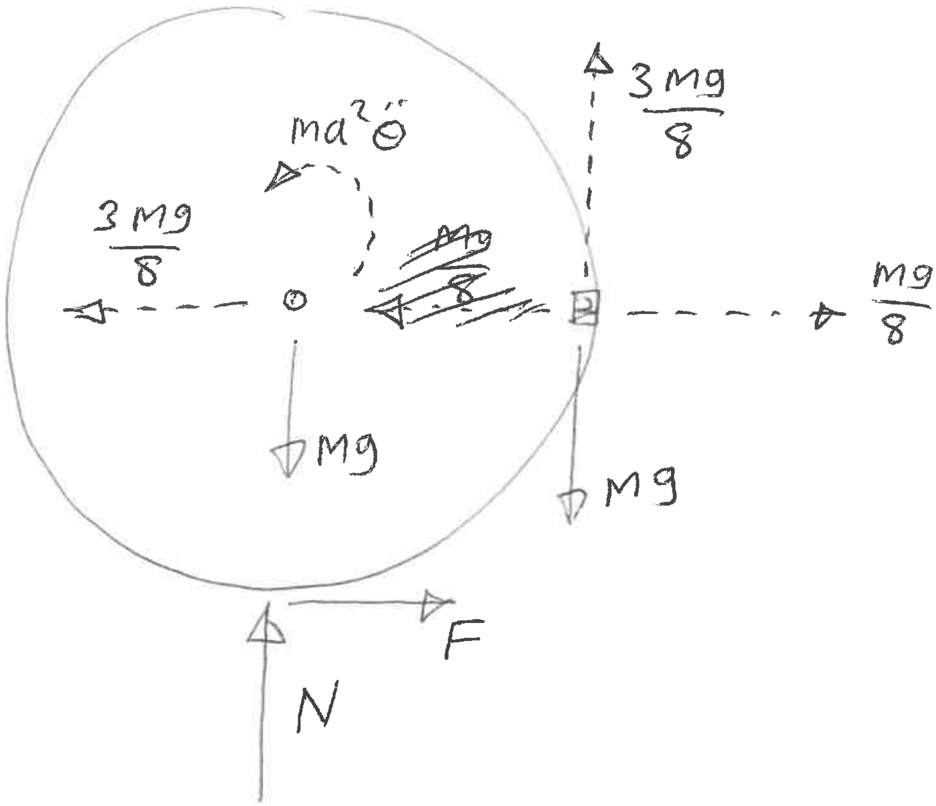
$$\dot{\theta}^2 = \frac{g}{2a}$$

$$\therefore \ddot{\underline{r}}_A = \frac{3g}{8}\,\underline{i}$$

$$\ddot{\underline{r}}_B = \left(\frac{3g}{8} - \frac{g}{2}\right)\underline{i} - \frac{3g}{8}\,\underline{j}$$

$$= -\frac{g}{8}\,\underline{i} - \frac{3g}{8}\,\underline{j}$$

d'Alembert




Resolve ∴ $N = 2mg - \frac{3mg}{8} = \frac{13\,mg}{8}$

$F = \frac{3mg}{8} - \frac{mg}{8} = \frac{mg}{4}$

$\mu = \frac{F}{N} = \frac{\frac{1}{4}}{13/8} = \frac{2}{13}$ //

ans

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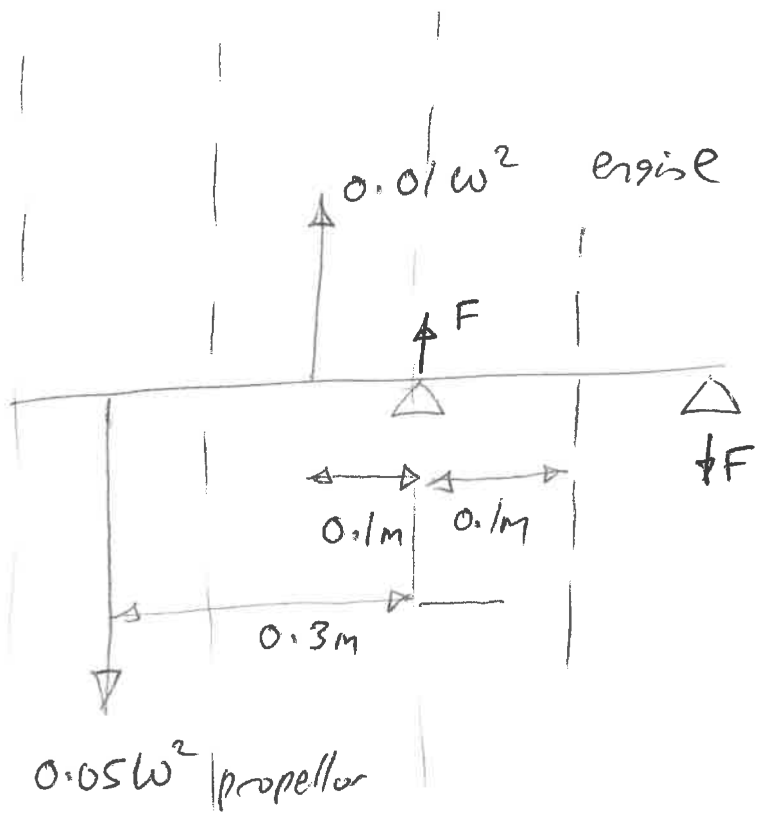
(a) Out of balance is minimized if they oppose, ie 180° offset

(i)


1250 rpm

$\omega \sim 125$ rad/s

(ii)




Moment on bearing

$$= (0.05 \times 0.4 - 0.01 \times 0.2)\omega^2$$

$$= 280 \text{ Nm}$$

(5)(b)(i) rolling creates no gyroscopic effect so the reaction torque in the bearing is unchanged from (a)(ii), 280 NM

(ii) Yaw results in a gyroscopic moment

$$Q = J \omega \Omega$$

$$J = \underset{\text{engine}}{\tfrac{1}{2} \times 180 \times 0.5^2} + \underset{\text{propellor}}{\tfrac{1}{2} \times 15 \times 1.3^2}$$

$$= 22.5 + 12.7$$

$$= 35.2 \quad kg\,m^2$$

Spin speed $\omega = 125$ rad/s as before

Yaw rate $\Omega = \frac{20}{180} \times \pi$ rad/s $= 0.349$ rad/s

$$Q = 35.2 \times 125 \times 0.349$$

$$= 1530 \quad Nm$$

(c) This gyroscopic effect is very strong and will cause the nose of the plane to lift (stall) or fall (dive) depending on direction of yaw

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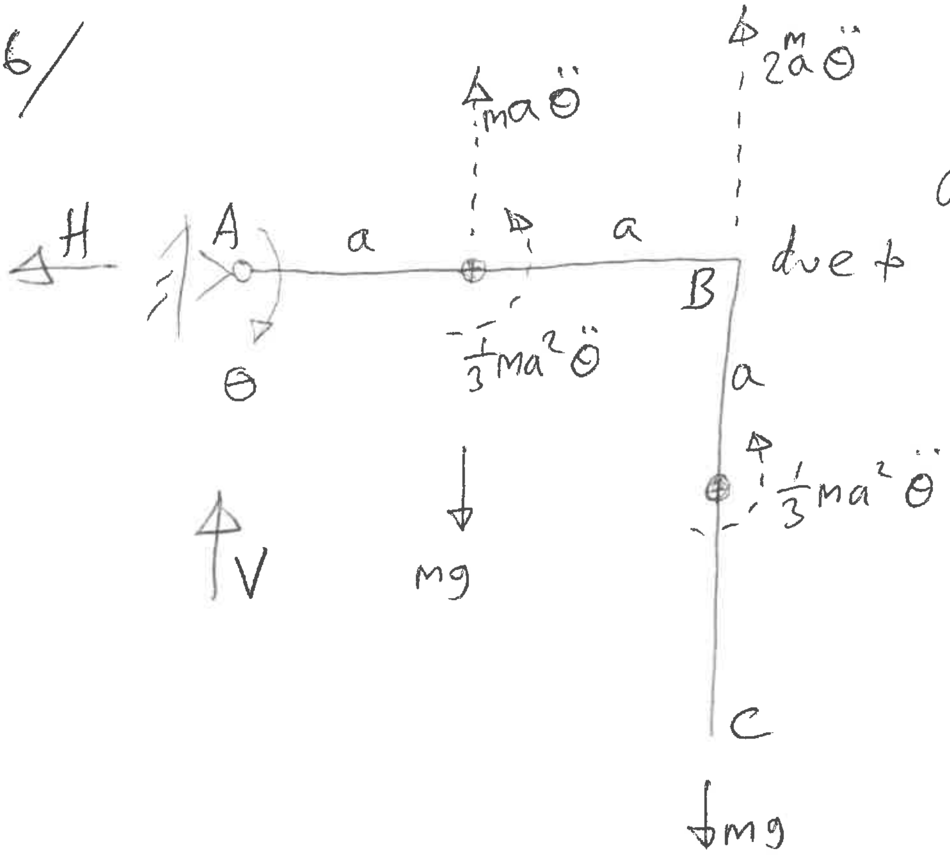
(a)




d'Alembert forces due to acceleration from rest

Take moments about A +

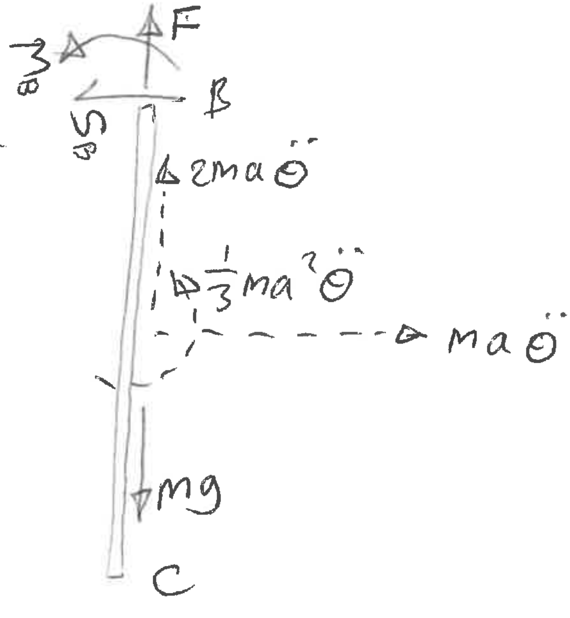
$$mga + mg\,2a = ma^2\ddot{\theta}\left(1 + \tfrac{1}{3} + 4 + 1 + \tfrac{1}{3}\right)$$

$$\therefore 3mga = \frac{20}{3}ma^2\ddot{\theta}$$

$$\therefore \ddot{\theta} = \frac{9}{20}\frac{g}{a} \quad \checkmark$$

ans

(b) FBD of bar BC



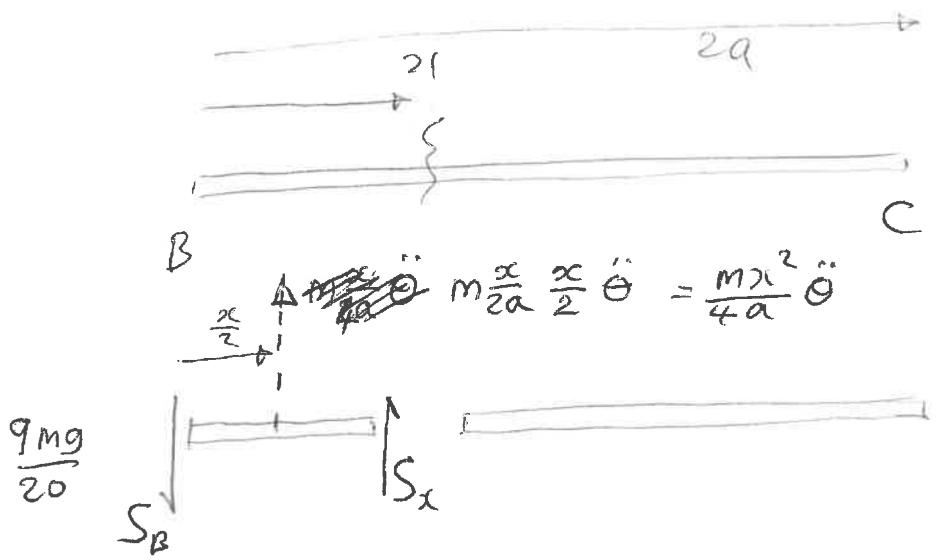
$$S_B = m a \ddot{\theta} = \frac{9mg}{20}$$ // ans

$$M = -\frac{1}{3} m a^2 \ddot{\theta} - m a^2 \ddot{\theta}$$

$$= -\frac{4}{3} m a^2 \frac{9g}{20a} = -\frac{3}{5} mga$$ // ans

(c)

FBD of BC at a distance $x$ from B

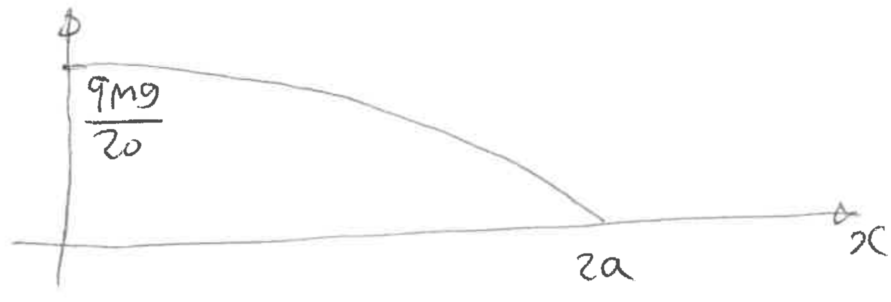



$$S_x = S_B - \frac{mx^2}{4a}\ddot{\theta} = \frac{9mg}{20} - \frac{mx^2}{4a}\frac{9g}{20a}$$

$$= \frac{9mg}{20}\left(1 - \frac{x^2}{4a^2}\right)$$

Note $S_a = 0$ at $x = 2a$ as expected

SFD

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