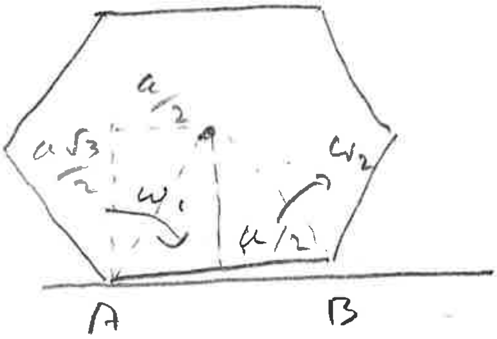
Paper 1 - Mechanics Solutions

1. a)



Moment of momentum conserved about B

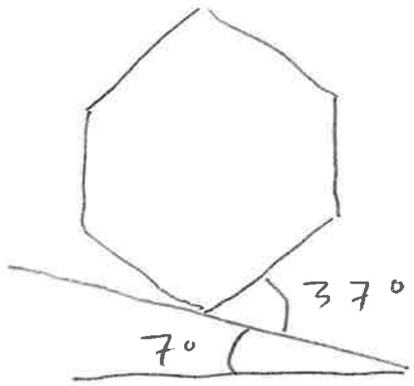
T.M.A.B, $\omega_1\left(k^2 + \frac{3a^2}{4} - \frac{a^2}{4}\right) = \omega_2\left(k^2 + a^2\right)$

For hexagon, $k^2 = \frac{a^2}{6}\left(2 + \frac{1}{2}\right) = \frac{5a^2}{12}$

$\therefore \omega_1 a^2\left(\frac{11}{12}\right) = \omega_2 a^2\left(\frac{17}{12}\right)$

$\omega_2 = \omega_1\left(\frac{11}{17}\right) \approx \underline{0.647\,\omega_1}$

b)



Rotates through $37°$ before first impact, so energy released is $mga(1 - \cos 37°)$

$KE = \frac{1}{2} m\omega_1^2\left(k^2 + a^2\right) = m\omega_1^2 a^2\left(\frac{17}{24}\right)$

$\therefore mga(0.2014) = m\omega_1^2 a^2(0.7083)$

So $\omega_1^2 = \frac{g}{a}\left(\frac{0.2014}{0.7083}\right) = \frac{g}{a}(0.2843)$

1 b) cont.

Using Part (a),

$$\omega_2^2 = \omega_1^2 \times 0.6471^2 = \frac{g}{a}(0.1990)$$

So energy after impact is

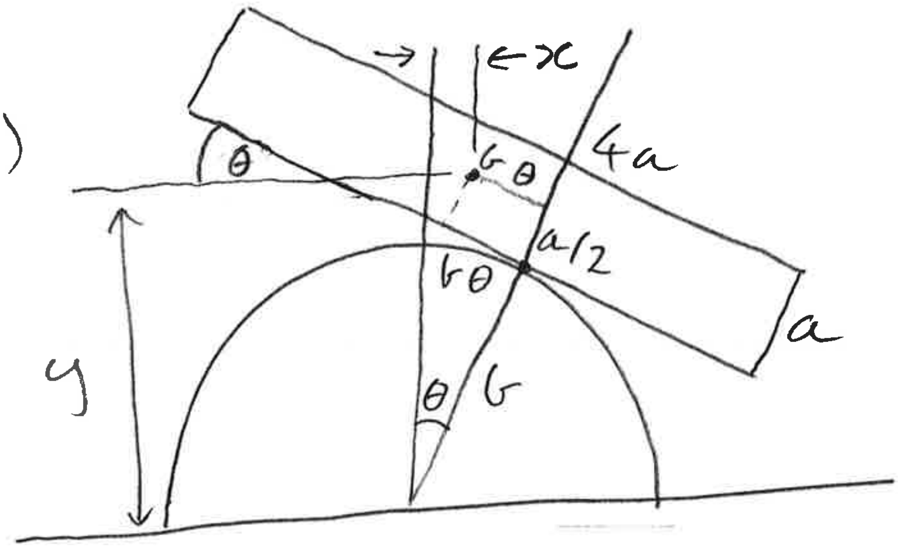
$$m\omega_2^2 a^2(0.7083) = mga(0.0843)$$

Energy needed to get to next peak is $mga(1 - \cos 23°)$

$$= 0.0795\, mga$$

So there is enough energy for the pencil to keep rolling.

2. a)



$$y = b\cos\theta + \frac{a}{2}\cos\theta + b\theta\sin\theta$$

and $\text{P.E.} = mgy$

$$\frac{dy}{d\theta} = b\sin\theta + b\theta\cos\theta - \left(b + \frac{a}{2}\right)\sin\theta$$
$$= b\theta\cos\theta - a/2\sin\theta$$

$$\frac{d^2y}{d\theta^2} = \left(b - \frac{a}{2}\right)\cos\theta - b\theta\sin\theta$$

When $\theta = 0$, $\cos\theta = 1$, $\sin\theta = 0$

So $\underline{b > a/2 \text{ for stability}}$

b) If $b = 2a$, then from Part (a)

$$V''(0) = mg\frac{3a}{2}$$

$$\text{K.E.} = \tfrac{1}{2}m(\dot{y}^2 + \dot{x}^2) + \tfrac{1}{2}I_G\dot{\theta}^2$$

Now $\dot{y} = \dfrac{dy}{d\theta}\dot{\theta} = 0$ when $\theta = 0$

and $x = \dfrac{5a}{2}\sin\theta - 2a\theta\cos\theta$

so $\dfrac{dx}{d\theta} = \dfrac{5a}{2}\cos\theta - 2a\cos\theta + 2a\theta\sin\theta$

2 (b) cont.

$\therefore$ at $\theta = 0$, $\dot{x} = \dfrac{dx}{d\theta}\dot{\theta} = \dfrac{a}{2}\dot{\theta}$

From data book, $k^2 = \dfrac{17a^2}{12}$

$$\therefore \text{K.E.} = \tfrac{1}{2} m\left(\frac{17a^2}{12} + \frac{a^2}{4}\right)\dot{\theta}^2$$

$$= \tfrac{1}{2}\underbrace{ma^2\left(\tfrac{5}{3}\right)}_{= M(\theta)}\dot{\theta}^2$$

Using the data book formula,

$$\omega_n^2 = \frac{\frac{3agm}{2}}{\frac{5ma^2}{3}} = \frac{9g}{10a}$$

So $\omega_n = 0.949\sqrt{\dfrac{g}{a}}$

3. a) Using Section 1.3 of the data book, and treating the housing as the body R:

Let Q be a point co-incident with P, but fixed to the housing

$$\underline{Q} = (-L\sin\theta,\ -d,\ L\cos\theta)$$

$$\underline{\omega} = (0,\ 0,\ \omega)$$

$$\left[\frac{d\underline{r}}{dt}\right]_R = (-\Omega L\cos\theta,\ 0,\ -\Omega L\sin\theta)$$

So $\underline{V}_Q = \underline{\omega} \times \underline{Q} = (\omega d,\ -\omega L\sin\theta,\ 0)$

and $\underline{V}_P = \underline{V}_Q + \left[\dfrac{d\underline{r}}{dt}\right]_R = \begin{bmatrix} \omega d - \Omega L\cos\theta \\ -\omega L\sin\theta \\ -\Omega L\sin\theta \end{bmatrix}$

b) $$\left[\frac{d^2 r}{dt^2}\right]_R = (-\Omega^2 L\sin\theta,\ 0,\ -\Omega^2 L\cos\theta)$$

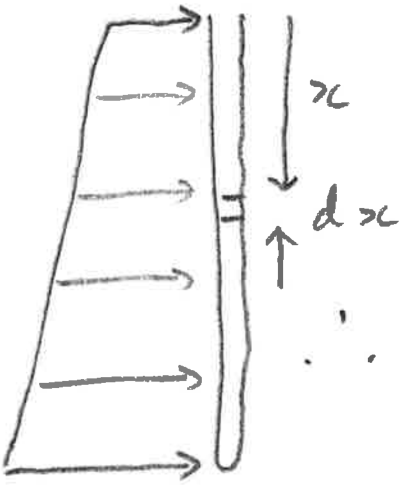
$$\underline{a}_Q = \underline{\omega} \times \underline{V}_Q = (\omega^2 L\sin\theta,\ \omega^2 d,\ 0)$$

$$2\underline{\omega} \times \left[\frac{d\underline{r}}{dt}\right]_R = (0,\ -2\omega\Omega L\cos\theta,\ 0)$$

Adding terms,

$$\underline{a}_P = \begin{bmatrix} \omega^2 + \Omega^2\ L\sin\theta \\ \omega^2 d - 2\omega\Omega L\cos\theta \\ -\Omega^2 L\cos\theta \end{bmatrix}$$

3. c) When $\theta = 180°$, P has no acceleration in the $\underline{i}$ direction, and acceleration in the $\underline{k}$ direction causes tension / compression — so bending only comes from acceleration in the $\underline{j}$ direction.



Acceleration x from root is $(d\omega^2 + 2\omega\Omega x)$

Mass/unit length is $\frac{m}{L}$

$$\therefore M = \int_0^L (d\omega^2 + 2\omega\Omega x)\, x \frac{m}{L}\, dx$$

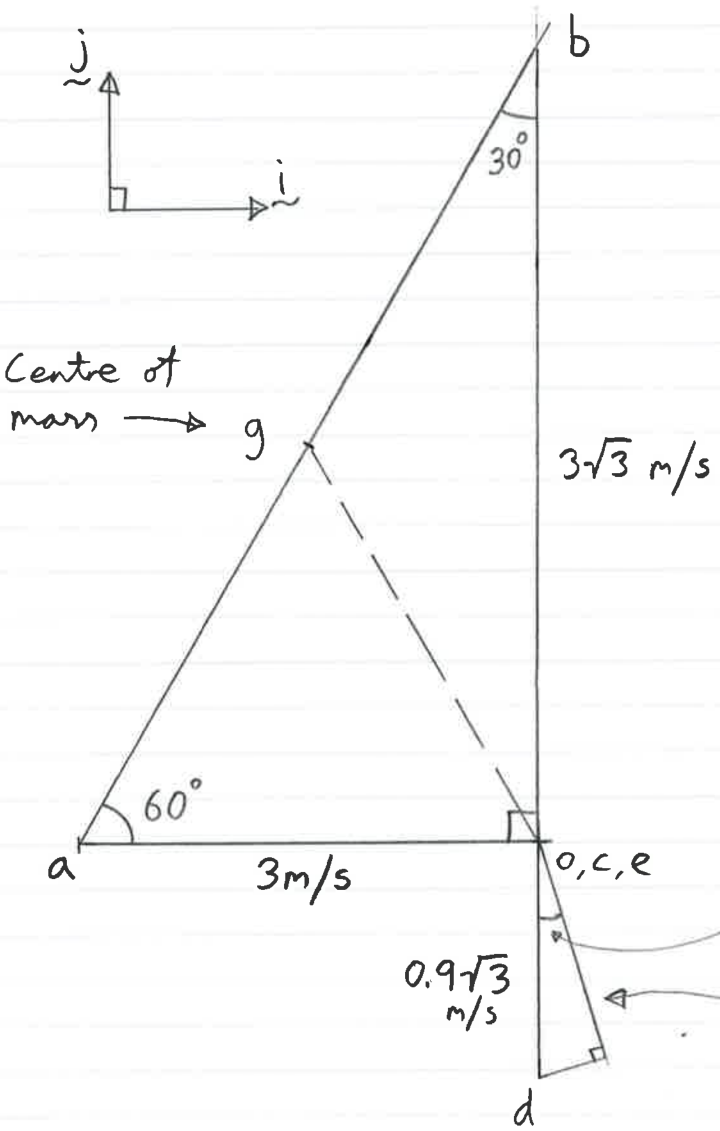
$$= \frac{m}{L}\left[\frac{d\omega^2 x^2}{2} + \frac{2\omega\Omega x^3}{3}\right]_{x=0}^{x=L}$$

$$\underline{M = \frac{m d \omega^2 L}{2} + \frac{2 m \omega \Omega L^2}{3}}$$

EGT1 Part IB Paper 1 - Mechanics

Section B

4) (a) Velocity diagram (scale 2 cm = 1m/s)



Centre of mass of door is G

$$\underset{\sim}{V}_G = -\frac{3}{2}\underset{\sim}{i} + \frac{3}{2}\sqrt{3}\underset{\sim}{j} \text{ m/s}$$

$$\omega_{AB} = \frac{ab}{AB} = \frac{6}{2} = 3 \text{ rad/s}$$

$$\omega_{BC} = \frac{bc}{BC} = \frac{3\sqrt{3}}{1} = 3\sqrt{3}$$

$$= 5.196 \text{ rad/s}$$

$\tan^{-1} 0.3 = 16.7°$

contraction rate of springs DE:

$$= 0.9\sqrt{3}\cos 16.7°$$

$$= \frac{0.9\sqrt{3}}{\sqrt{1.09}}$$

$$= 1.493 \text{ m/s}$$

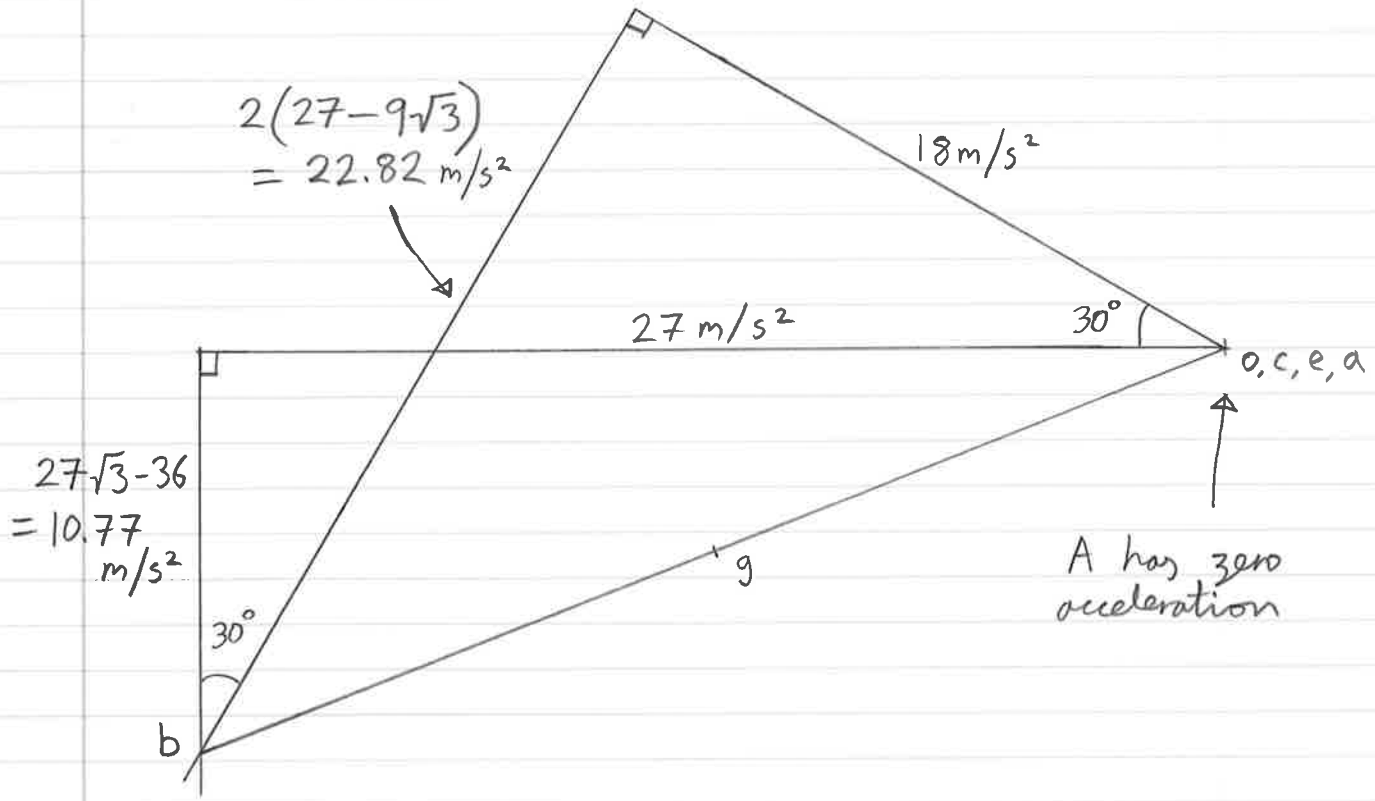
4) (b) (i)

Centripetal acceleration components:

$$|AB|\,\omega_{AB}^2 = 2 \times 3^2 = 18 \text{ m/s}^2$$

$$|BC|\,\omega_{BC}^2 = 1 \times (3\sqrt{3})^2 = 27 \text{ m/s}^2$$

Acceleration diagram (scale 1cm = 2m/s²)



Acceleration of the centre of mass

$$\underset{\sim}{a}_G = -13.5\,\underset{\sim}{i} - (13.5\sqrt{3} - 18)\,\underset{\sim}{j}$$

$$= -13.5\,\underset{\sim}{i} - 5.383\,\underset{\sim}{j} \quad \text{m/s}^2$$

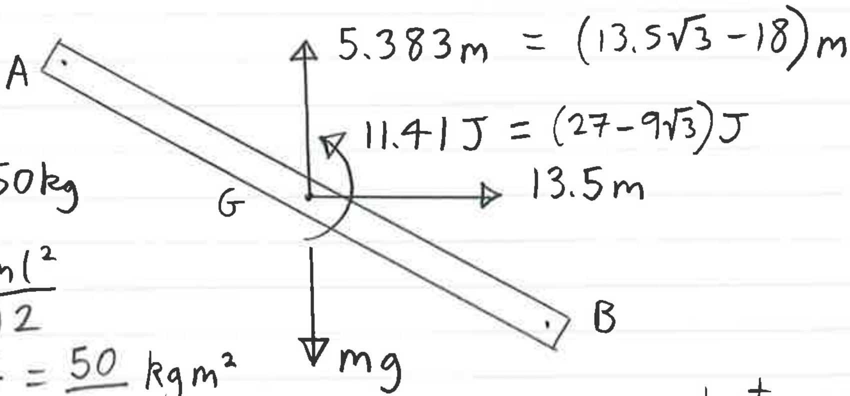
Angular acceleration of AB

$$-|AB|\,\dot{\omega}_{AB} = 2(27 - 9\sqrt{3})$$

$$\therefore \dot{\omega}_{AB} = 9\sqrt{3} - 27 = -11.41 \text{ rad/s}^2 ↻$$

(i.e. 11.41 rad/s² ↺)

(b) (ii) D'Alembert forces and torque:



where $m = 50\text{kg}$

and $J = \dfrac{ml^2}{12}$

$= \dfrac{50 \times 2^2}{12} = \dfrac{50}{3}\ \text{kgm}^2$

Let $g = 9.81\,\text{m/s}^2$

Power:

Work done by: F + self weight + d'Alembert forces + torque + springs $= 0$

Thus:

$$3\sqrt{3}F - \frac{3}{2}\sqrt{3}mg - \frac{3}{2} \times 13.5m + \frac{3}{2}\sqrt{3} \times (13.5\sqrt{3} - 18)m$$

$$+ 3 \times (27 - 9\sqrt{3})\frac{ml^2}{12} + \frac{0.9\sqrt{3}}{\sqrt{1.09}} \times 600 = 0$$

$$\therefore F = \frac{1}{2}mg + \frac{13.5m}{2\sqrt{3}} - \frac{1}{2}(13.5\sqrt{3} - 18)m$$

$$- \frac{1}{\sqrt{3}}(27 - 9\sqrt{3})\frac{ml^2}{12} - \frac{0.3}{\sqrt{1.09}} \times 600$$

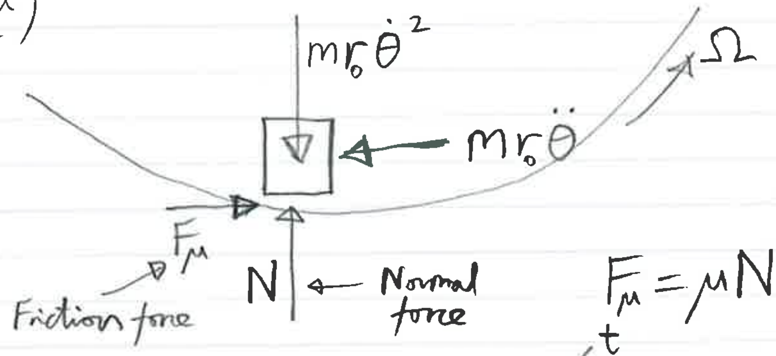
Note that springs help!

$$= 245.25 + 194.86 - 134.57$$

$$- 109.81 - 172.41$$

$$= 23.3 \text{ N}$$

5) (a)



Initially $\dot{\theta}$ is zero

$F_\mu = mr_0\ddot{\theta}$ $\quad \therefore \ddot{\theta} = \mu\dot{\theta}^2$

$N = mr_0\dot{\theta}^2$

$F_\mu = \mu N = \mu m r_0 \dot{\theta}^2$

$$\text{Work done by centrifuge} = \int_0^t \underbrace{F_\mu r_0 \Omega}_{\text{power}}\, dt = \int_0^t \mu m r_0 \dot{\theta}^2 r_0 \Omega \, dt$$

$$= \mu m r_0^2 \Omega \int_0^t \left(\frac{d\theta}{dt}\right)^2 dt = \mu m r_0^2 \Omega \int_0^t \frac{1}{\mu}\frac{d^2\theta}{dt^2}\, dt =$$

$$= mr_0^2\Omega \int_0^\Omega d\left(\frac{d\theta}{dt}\right) = mr_0^2 \left[\frac{d\theta}{dt}\right]_0^\Omega = mr_0^2\Omega$$

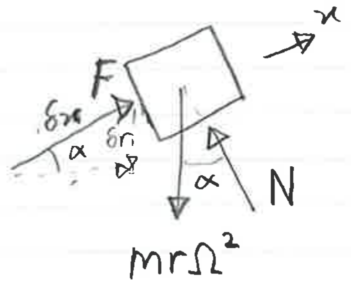
or put more simply:

$$\text{Work done by centrifuge} = \int_0^t F_\mu r_0 \Omega \, dt = \int_0^t mr_0\ddot{\theta}\, r_0\Omega\, dt$$

$$= mr_0^2\Omega \int_0^t \ddot{\theta}\, dt = mr_0^2\Omega^2 \quad (\text{final } \dot{\theta} \text{ is } \Omega)$$

(b)(i) Work done by F $= \displaystyle\int_{r_0}^{r_i} \underset{\sim}{F}\cdot d\underset{\sim}{r} = \int_{x=0\,(r_0)}^{x_i\,(r_i)} F\, dx$

$\delta r = -\sin\alpha\, \delta x$



$F = mr\Omega^2 \sin\alpha$

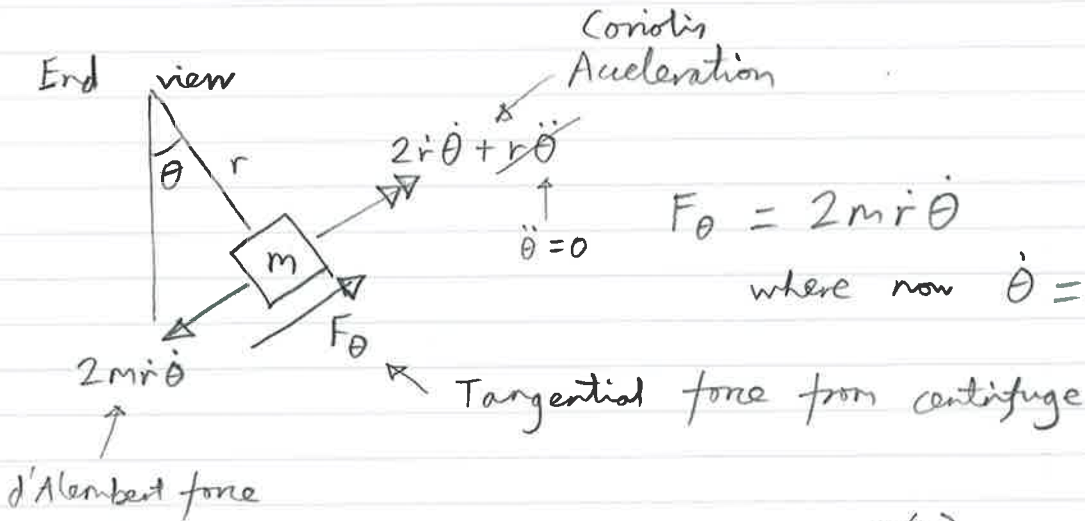
5) (b) (i) (cont.)

$$\therefore \text{ Work done by } F = \int_{r_o}^{r_i} m r \Omega^2 \sin\alpha \cdot \frac{dr}{-\sin\alpha}$$

$$= -m\Omega^2 \int_{r_o}^{r_i} r \, dr = -m\Omega^2 \left[\frac{r^2}{2}\right]_{r_o}^{r_i}$$

$$= -\frac{1}{2} m\Omega^2 \{r_i^2 - r_o^2\} = \frac{1}{2} m (r_o^2 - r_i^2)\Omega^2$$

(ii) End view



$$F_\theta = 2m\dot{r}\dot{\theta}$$

where now $\dot{\theta} = \Omega$

$$\text{Work done by centrifuge (drum)} = \int_{\theta=0\,(r_o)}^{\theta(r_i)} F_\theta \, r \, d\theta$$

$$= \int_{\theta=0\,(r_o)}^{\theta(r_i)} 2m \frac{dr}{dt} \frac{d\theta}{dt} r \, d\theta$$

$$= 2m \int_{r_o}^{r_i} \left(\frac{d\theta}{dt}\right)^2 r \, dr = 2m \int_{r_o}^{r_i} \Omega^2 r \, dr = 2m\Omega^2 \left[\frac{r^2}{2}\right]_{r_o}^{r_i}$$

$$= m\Omega^2 (r_i^2 - r_o^2)$$

$$= -m\Omega^2 (r_o^2 - r_i^2)$$ i.e. the work done is negative

5) (b) (ii) (cont.)

Alternatively consider kinetic energy as m is pushed from r_0 to r_i:

Initial K.E. $= \frac{1}{2} m (r_0 \Omega)^2$

Final K.E. $= \frac{1}{2} m (r_i \Omega^2)$

$\therefore$ Total work done on m (in conical section) is $\frac{1}{2} m (r_i^2 - r_0^2) \Omega^2$

$= -\frac{1}{2} m (r_0^2 - r_i^2) \Omega^2$

i.e. negative

And total work done (in conical section) = Work done by F + Work done by centrifuge (F_θ)

$\therefore$ Work done by centrifuge (F_θ) = Total W.D. (conical) − W.D. by F

$= -\frac{1}{2} m (r_0^2 - r_i^2) \Omega^2 - \frac{1}{2} m (r_0^2 - r_i^2) \Omega^2$

$= -m (r_0^2 - r_i^2) \Omega^2$

(c) Total work done to process a mass m

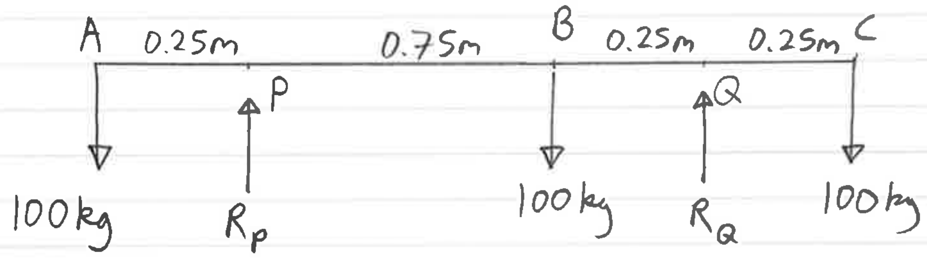
= Acceleration work (part (a)) + Work done by F (b)(i) + Work done by F_θ (b)(ii)

$= m r_0^2 \Omega^2 + \frac{1}{2} m (r_0^2 - r_i^2) \Omega^2 - m (r_0^2 - r_i^2) \Omega^2$

$= m r_0^2 \Omega^2 - \frac{1}{2} m (r_0^2 - r_i^2) \Omega^2$

$= \frac{1}{2} m (r_0^2 + r_i^2) \Omega^2$

6) (a) (i)



Assume aircraft rolls about same longitudinal axis as engine.

Thus only loads on bearings P and Q are due to self-weight

$P\circlearrowright \quad 1 \times R_Q = 1.25 \times 100 + 0.75 \times 100 - 0.25 \times 100$

$\therefore \quad R_Q = (1.25 + 0.75 - 0.25) \times 100$

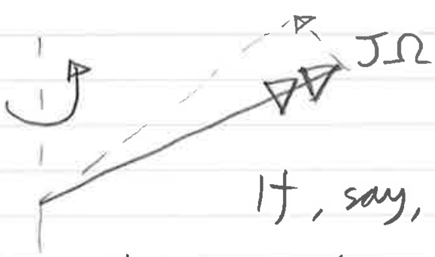
$= 175 \text{ kgf} = 1717 \text{ N} \quad (g = 9.81 \text{ m/s}^2)$

and $\quad R_P = 125 \text{ kgf} = 1226 \text{ N}$

(ii) $\quad J = 3 \times \frac{1}{2} m r^2 = \frac{3}{2} \times 100 \times 0.3^2 = 13.5 \text{ kgm}^2$

Gyroscopic torque $\quad T = J \Omega \omega$

$= 13.5 \times 10{,}000 \times \frac{2\pi}{60} \times 2 \times \frac{2\pi}{60}$



$= 2961 \text{ Nm}$

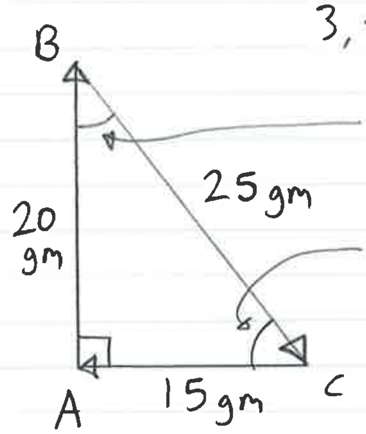
If, say, aircraft turns left and engine spins clockwise (when looking from C towards A) the required gyro torque T is provided by an increase in R_P and a decrease in R_Q of 2961 N.

If turn or spin are in the opposite direction then the effect on R_P & R_Q is reversed, but still vertical.

6) (b)

(i)

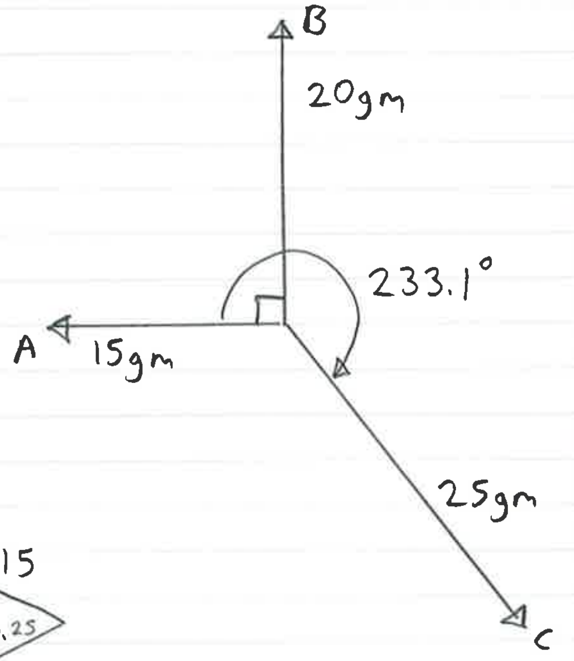
3, 4, 5 triangle



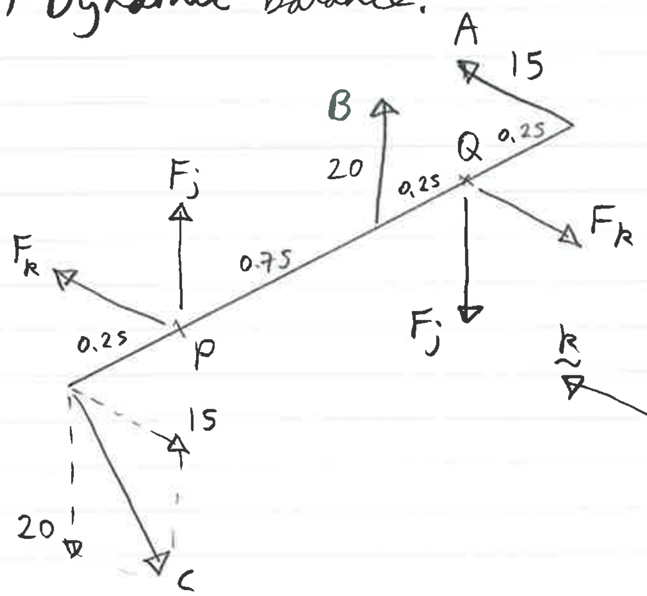
$\sin^{-1} \frac{3}{5} = 36.9^\circ$

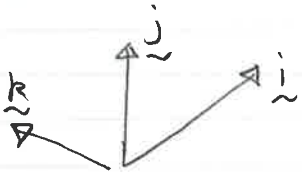
$\cos^{-1} \frac{3}{5} = 53.1^\circ$

Orientations for static balance.



(ii) Dynamic balance:





$\Omega = 10{,}000\,\text{rpm} = 1047\ \text{rad/s}$

Moments about $\underset{\sim}{j}$ axis @ P:

$$(15 \times 0.25 + 15 \times 1.25) \times 10^{-3} \Omega^2 = F_k \times 1 \quad \therefore F_k = 15 \times 1.5 \times 10^{-3} \Omega^2$$
$$= 24674\,\text{N}$$

Moments about $\underset{\sim}{k}$ axis @ P:

$$(20 \times 0.75 + 20 \times 0.25) \times 10^{-3} \Omega^2 = F_j \times 1 \quad \therefore F_j = 20 \times 10^{-3} \Omega^2$$
$$= 21932\ \text{N}$$

Thus total $F = \sqrt{F_j^2 + F_k^2} = 33.0\,\text{kN}$

inclined @ 41.6° to out-of balance A